Interpolation failures in semilinear substructural logics

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Introduction

Bridge theorems are among the most interesting results in the field of algebraic logic, connecting logical (and mostly syntactic) features of deductive systems and properties of classes of algebras.

One of the most interesting ones states the connection between the **deductive interpolation property** of a logic and the **amalgamation property** of the corresponding variety of algebras.

Introduction

We say that a logic L, associated to an equivalence relation $\vdash$, has the **deductive interpolation property** if for any set of formulas $\Gamma \cup \{\psi\}$ over the appropriate language, if $\Gamma \vdash \psi$ then there exists a formula δ such that $\Gamma \vdash \delta$, $\delta \vdash \psi$ and $Var(\delta) \subseteq Var(\Gamma) \cap Var(\psi)$.

The **interpolation property** of a (strongly algebraizable) logic corresponds to the **amalgamation property** of the corresponding variety of algebras, under the assumption that such variety has the congruence extension property (CEP).

Introduction

In this work we show that the following varieties do not have the amalgamation property:

- semilinear commutative residuated lattices;
- semilinear commutative integral residuated lattices;
- MTL-algebras and its pseudocomplemented and involutive subvarieties (SMTL and IMTL respectively);
- the n-potent subvariety of each of the previous varieties for $n \geq 2$.

Introduction

Let us then recall some definitions and results that will be useful later on. A **commutative residuated lattice**, or CRL, is a structure $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ where:

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a commutative monoid,
- $x \cdot y \leq z$ iff $y \leq x \to z$

CRLs constitute a variety, CRL.

Introduction

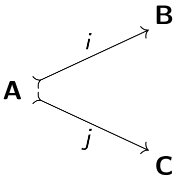
We call a commutative residuated lattice:

- **integral** if 1 is the top element of the lattice, and write CIRL
- **bounded** if integral and there is an extra constant 0 in the signature that is the least element of the lattice. We write BCIRLs for bounded commutative integral residuated lattices.

We call **chain** a totally ordered structure. The variety of BCIRLs generated by chains is **MTL**, the algebraic counterpart of the logic MTL. Moreover,

- **IMTL**: MTL-algebras + involutivity, i.e. $\neg\neg x = x$;
- **SMTL**: MTL-algebras + pseudocomplementation, i.e. $x \wedge \neg x = 0$.

Let K be a class of algebras in the same signature.



V-formation

Introduction

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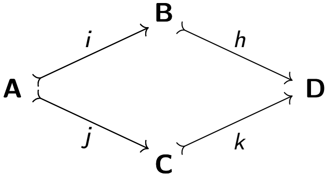

Amalgam

Introduction

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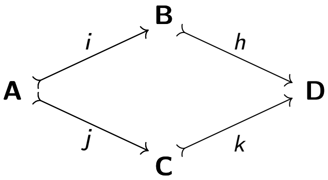

Amalgam

In particular, K has the **amalgamation property (AP)** if each V-formation in K has an amalgam in K.

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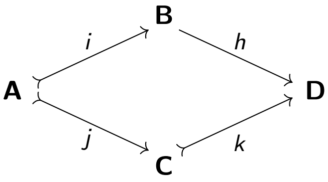

1-Amalgam

Introduction

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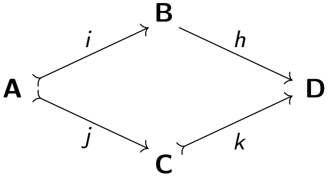


1-Amalgam

In particular, K has the **one-sided amalgamation property (1AP)** if each V-formation in K has a 1-amalgam in K.

Introduction

Theorem (Fussner-Metcalfe)

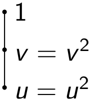
Let V *be a variety with the CEP and such that* V_{FSI} *is closed under subalgebras. The following are equivalent:*

1. V *has the amalgamation property;*
2. V_{FSI} *has the one-sided amalgamation property.*

This result is particularly useful in varieties generated by commutative residuated chains; indeed, all commutative residuated lattices have the CEP and a semilinear residuated lattice is finitely subdirectly irreducible if and only if it is totally ordered.

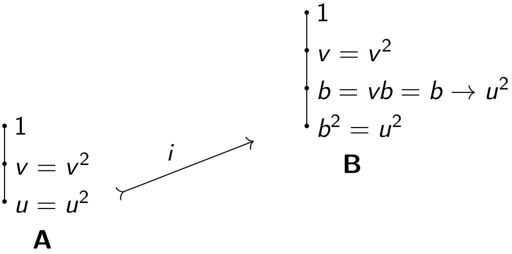
Amalgamation failure

The goal is to find a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f, g)$, which we will call $\mathcal{VS}$-**formation**, that does not have an amalgam in chains. As usual, one can consider f, g to be the identity, i.e. **A** is a subalgebra of both **B** and **C**.

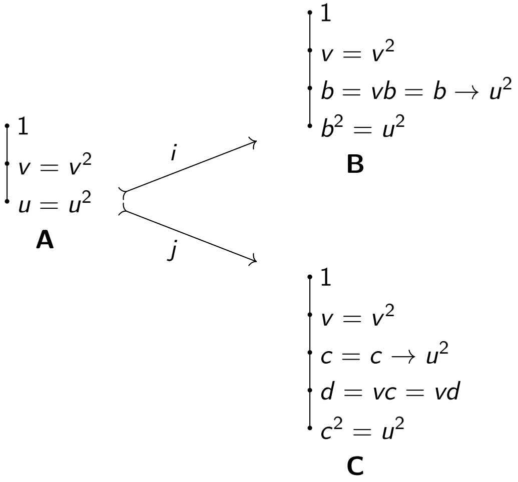


A

Amalgamation failure



Amalgamation failure



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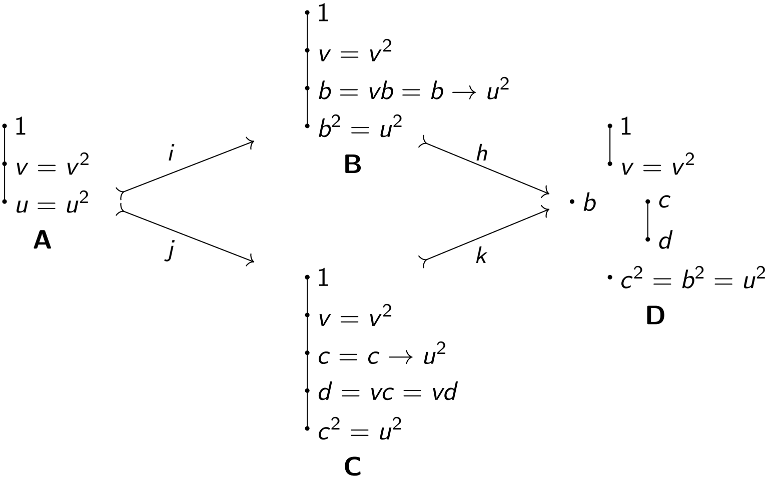

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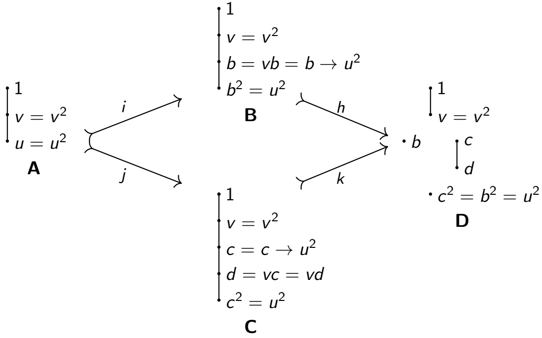
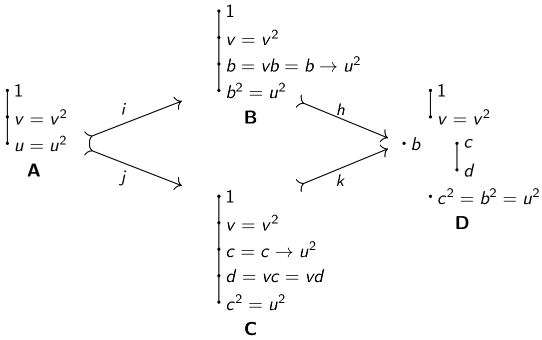


If $b < c$, then, by order preservation, $bc \leq c^2 = u^2$ and so $c \leq b \to u^2 = b$, **CONTRADICTION!**

Amalgamation failure



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Amalgamation failure

Theorem

There is no amalgam $(\mathbf{D}, h, k)$ *to* $\mathcal{VS}$ *in the class of RL-chains.*

Interestingly the same V-formation yields the failure of the one-sided amalgamation in chains. Indeed it can be shown that any 1-amalgam would have to be an amalgam.

Theorem

There is no one-amalgam $(\mathbf{D}, h, k)$ *to* $\mathcal{VS}$ *in the class of RL-chains.*

Amalgamation failure

Hence,

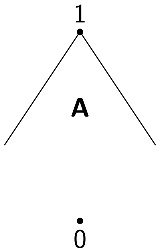
Theorem

Let V *be a variety of semilinear RLs (or FL-algebras) with the congruence extension property and such that the algebras* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *in* $\mathcal{VS}$ *belong to* V. *Then* V *does not have the AP.*

Observe that a variety of residuated lattices does not have to be commutative and it can be unbounded to satisfy the hypothesis of the previous theorem.

Lifting and rotation: general idea

Given an integral residuated lattice **A**,



$\mathbf{2} \oplus \mathbf{A}$

Lifting and rotation: general idea

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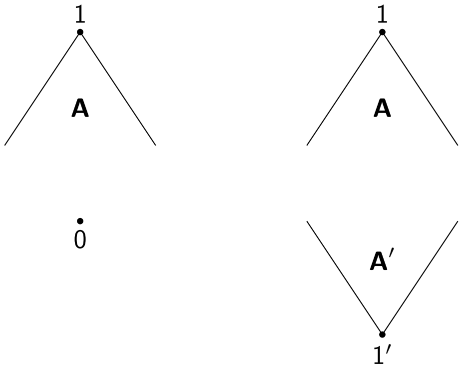
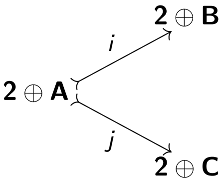


$\mathbf{2} \oplus \mathbf{A}$ $\quad$ $\mathbf{A}^{\delta}$

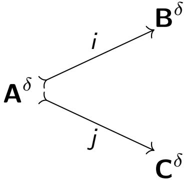
Let us consider again the algebras involved in the $\mathcal{VS}$-formation.



Then, we still cannot find a 1-amalgam (or amalgam) for such V-formation, denoted by $\mathcal{VS}^2$, since we will find exactly the same contradiction.

Amalgamation failure in IMTL

Let us consider again the algebras involved in the $\mathcal{VS}$-formation.



Then, we still cannot find a 1-amalgam (or amalgam) for such V-formation, which we denote with $\mathcal{VS}^\delta$, since we will find exactly the same contradiction.

Amalgamation failure in IMTL and SMTL algebras

Hence,

Theorem

Let V *be a variety of IMTL-algebras (SMTL-algebras) and such that the algebras* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *in* $\mathcal{VS}^2$ *(in* $\mathcal{VS}^\delta$*) belong to* V*. Then* V *does not have the AP.*

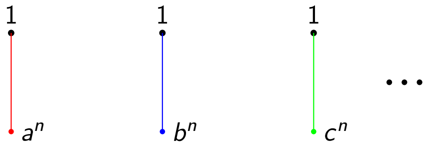
Actually, this result holds in a more general setting.

Theorem

Let V *be a variety of semilinear residuated lattices with the congruence extension property, and such that the algebras* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *in* $\mathcal{VS}^\delta$ *(*$\mathcal{VS}^2$*) belong to* V*. Then* V *does not have the AP.*

Constructing other counterexamples

Starting from the partial gluing (introduced by Galatos and Ugolini), we introduced its iterated version. Pictorially,



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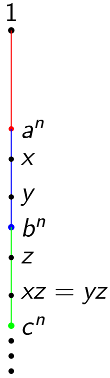

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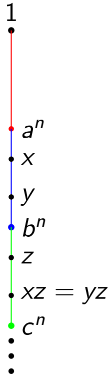

With this construction one can find countably many failures of the amalgamation property since the algebras in the $\mathcal{VS}$-formation can be constructed as partial gluings.

The varieties B_n

For any $n \in \mathbb{N}$, let B_n the variety generated by partial gluing of simple, n-potent chains. We can provide an axiomatization for such variety:

1. CIRL with prelinearity;
2. $x^n = x^{n+1}$;
3. $(x^n \to (x^n \wedge y^n)) \vee ((x(x \wedge y) \to x^n(x \wedge y)) \wedge ((x \wedge y)x \to (x \wedge y)x^n)) = 1$

The $\bigvee_{n\in\mathbb{N}} \mathsf{B}_n$

One can consider now the variety generated by the join of all B_n, $\bigvee_{n\in\mathbb{N}} \mathsf{B}_n$.

Let then PGA be the class of chains which are partial gluing of their archimedean components.

Theorem

$\mathsf{V}(\mathsf{PGA}) = \bigvee_{n\in\mathbb{N}} \mathsf{B}_n$.

Observe that $\bigvee_{n\in\mathbb{N}} \mathsf{B}_n \subseteq \mathsf{V}(\mathsf{PGA})$ since, for any $n \in \mathbb{N}$ every chain $\mathbf{A} \in \mathsf{B}_n$ is in PGA. For the converse, we show that given $\mathbf{C} \in \mathsf{PGA}$, each of its partial subalgebras embeds into some chain $\mathbf{B} \in \mathsf{B}_n$ for some $n \in \mathbb{N}$.

The $\bigvee_{n\in\mathbb{N}} \mathsf{B}_n$

Observe also that the variety $\bigvee_{n\in\mathbb{N}} \mathsf{B}_n \subseteq MTL$, hence one can wonder if the two varieties coincides.

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Theorem

$\bigvee_{n\in\mathbb{N}} \mathsf{B}_n \neq MTL$.

We can prove that for any finite chain $\mathbf{A}$, $\mathbf{A} \in \bigvee_{n\in\mathbb{N}} \mathsf{B}_n$ if and only if $\mathbf{A} \in \mathsf{B}_n$ for some $n \in \mathbb{N}$.

Conclusions

Theorem

The following varieties do not have the AP:

1. MTL
2. *n-potent* MTL*-algebras for* $n \geq 2$
3. IMTL
4. SMTL
5. B_n *for* $n \geq 2$.

and their unbounded versions:

1. SemCIRL
2. SemCRL *(Fussner-Santchi)*
3. $SemFL_e$*(Fussner-Santchi)*

Thanks for your attention