Decidability and undecidability in substructural logics

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BLAST, Boulder, May 2025

Outline

- Equational decidability for lattices (using Algebraic Proof Theory)
- Residuated lattices and Substructural Logics
- Undecidability via Counter Machines
- Undecidability via Projective Geometry, Tag systems, Wang tilings.
- Decidability via residuated frames
- Decidability via diagrams
- Complexities.

Equational decidability for lattices

Given an equation $s = t$ in the language of lattices, decide whether it holds in all lattices.

The class of lattices is defined by equations, so we can use first-order logic to check if $s = t$ is derivable from the axioms: $\wedge$ and $\vee$ are associative, commutative and idempotent, and the absorption laws hold: $x \vee (y \wedge x) = x = x \wedge (y \vee x)$.

We can do a bit better. We can use the equational-logic fragment of FO logic which uses the axioms and the rules

$$\frac{}{s = s}\ (ref) \qquad \frac{s = t}{t = s}\ (sym) \qquad \frac{s = t \quad t = r}{s = r}\ (tr)$$

$$\frac{s = r \quad t = u}{s \vee t = r \vee u}\ (Cong\vee) \qquad \frac{s = r \quad t = u}{s \wedge t = r \wedge u}\ (Cong\wedge) \qquad \frac{s = t}{\sigma(s) = \sigma(t)}\ (FI_\sigma)_{\sigma \in End(Tm)}$$

Lattices are definitionally equivalent to *lattice-ordered posets*, so we could use inequalities:

$$\frac{}{s \leqslant s}\ (ref) \qquad \frac{s \leqslant t \quad t \leqslant s}{s = t}\ (as) \qquad \frac{s \leqslant t \quad t \leqslant r}{s \leqslant r}\ (tr)$$

$$\frac{}{s \leqslant s \vee t}\ (R\vee_1) \qquad \frac{}{t \leqslant s \vee t}\ (R\vee_2) \qquad \frac{s \leqslant r \quad t \leqslant r}{s \vee t \leqslant r}\ (L\vee)$$

$$\frac{}{s \wedge t \leqslant s}\ (L\wedge_1) \qquad \frac{}{s \wedge t \leqslant t}\ (L\wedge_2) \qquad \frac{r \leqslant s \quad r \leqslant t}{r \leqslant s \wedge t}\ (R\wedge)$$

A better system

The slightly better system Lat:

$$\frac{}{s \leqslant s}\,(ref) \qquad \frac{s \leqslant t \quad t \leqslant s}{s = t}\,(as) \qquad \frac{s \leqslant t \quad t \leqslant r}{s \leqslant r}\,(tr)$$

$$\frac{r \leqslant s}{r \leqslant s \vee t}\,(R\vee_1) \qquad \frac{r \leqslant t}{r \leqslant s \vee t}\,(R\vee_2) \qquad \frac{s \leqslant r \quad t \leqslant r}{s \vee t \leqslant r}\,(L\vee)$$

$$\frac{s \leqslant r}{s \wedge t \leqslant r}\,(L\wedge_1) \qquad \frac{t \leqslant r}{s \wedge t \leqslant r}\,(L\wedge_2) \qquad \frac{r \leqslant s \quad r \leqslant t}{r \leqslant s \wedge t}\,(R\wedge)$$

The following is a *derivation* in Lat of the equation $x \vee y = y \vee x$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \leqslant x}\,(ref)}{x \leqslant y \vee x}\,(R\vee) \quad \dfrac{\dfrac{}{y \leqslant y}\,(ref)}{y \leqslant y \vee x}\,(R\vee)}{x \vee y \leqslant y \vee x}\,(L\vee) \qquad \dfrac{\dfrac{\dfrac{}{y \leqslant y}\,(ref)}{y \leqslant x \vee y}\,(R\vee) \quad \dfrac{\dfrac{}{x \leqslant x}\,(ref)}{x \leqslant x \vee y}\,(R\vee)}{y \vee x \leqslant x \vee y}\,(L\vee)}{x \vee y = y \vee x}\,(as)$$

The rule of transitivity is redundant! So, a naive inverse-proof search yields decidability of equations in lattices. (This gives *proof theory* for lattices.)

To show that the system Lat^- (Lat without tranitivity) proves the same equations, i.e., that if an equation is not provable in Lat^- then it fails in lattices, we need a way to construct (counterexample) lattices.

Birkhoff's polarities aka Formal Concept Analysis

Let N be the binary relation on the set Tm of lattice terms, defined by:

$$s\ N\ t \quad \text{iff} \quad s \leqslant t \text{ is derivable in } Lat^{-}.$$

For $S, T \subseteq Tm$ we write $S\ N\ T$ for: $s\ N\ t$, for all $s \in S$ and $t \in T$. Also,

$$S^{\rhd} = \{r \in Tm : S\ N\ \{r\}\} \text{ and } S^{\lhd} = \{r \in Tm : \{r\}\ N\ S\}.$$

The map γ on the powerset $\mathcal{P}(Tm)$ defined by $\gamma(S) = S^{\rhd\lhd}$ is then a closure operator and $\mathbf{Tm}^{+} := (\gamma[\mathcal{P}(Tm)], \cap, \cup_{\gamma})$ forms a complete lattice, where $S \cup_{\gamma} T = \gamma(S \cup T)$.

Also, the sets $\{r\}^{\lhd}$, $r \in Tm$, *form a basis* for γ: for every set $R \subseteq Tm$, we have $R^{\lhd} = (\bigcup_{r \in R}\{r\})^{\lhd} = \bigcap_{r \in R}\{r\}^{\lhd}$. (The maps $^{\rhd}$ and $^{\lhd}$ form a *Galois connection*.)

Lemma 1. For every s, t in Tm and every $S, T \in \gamma[\mathcal{P}(Tm)]$, if $s \in S \subseteq \{s\}^{\lhd}$ and $t \in T \subseteq \{t\}^{\lhd}$, then $s \wedge t \in S \cap T \subseteq \{s \wedge t\}^{\lhd}$ and $s \vee t \in S \cup_{\gamma} T \subseteq \{s \vee t\}^{\lhd}$.

Proof. To show $S \cap T \subseteq \{s \wedge t\}^{\lhd}$, let $r \in S \cap T \subseteq \{s\}^{\lhd} \cap \{t\}^{\lhd}$. Then $r\ N\ s$ and $r\ N\ t$, so by $(R\wedge)$ we get $r\ N\ s \wedge t$, hence $r \in \{s \wedge t\}^{\lhd}$.

For $s \wedge t \in S \cap T$, we will show $s \wedge t \in S$, as we can get $s \wedge t \in T$ is a similar way. Since the sets $\{r\}^{\lhd}$, $r \in Tm$, form a basis, it suffices to show: $S \subseteq \{r\}^{\lhd} \Longrightarrow s \wedge t \in \{r\}^{\lhd}$. Since $s \in S$, we have $s \in \{r\}^{\lhd}$; so $s\ N\ r$. From $(L\wedge)$ we get $s \wedge t\ N\ r$, hence $s \wedge t \in \{r\}^{\lhd}$.

Transitivity elimination

Lemma 1. For every s, t in Tm and every $S, T \in \gamma[\mathcal{P}(Tm)]$, if $s \in S \subseteq \{s\}^{\lhd}$ and $t \in T \subseteq \{t\}^{\lhd}$, then $s \wedge t \in S \cap T \subseteq \{s \wedge t\}^{\lhd}$ and $s \vee t \in S \cup_{\gamma} T \subseteq \{s \vee t\}^{\lhd}$.

Proof (cont). $S \cup_{\gamma} T \subseteq \{s \vee t\}^{\lhd}$: we first show that $S \subseteq \{s \vee t\}^{\lhd}$. For $r \in S$, we have $r \in S \subseteq \{s\}^{\lhd}$, so $r \; N \; s$. By $(R\vee_1)$ we have $r \; N \; s \vee t$, namely $r \in \{s \vee t\}^{\lhd}$. Likewise we have $T \subseteq \{s \vee t\}^{\lhd}$, so $S \cup T \subseteq \{s \vee t\}^{\lhd}$. By applying γ on both sides and using that $R^{\lhd\rhd\lhd} = R^{\lhd}$, for all $R \subseteq Tm$, we get $S \cup_{\gamma} T \subseteq \{s \vee t\}^{\lhd}$.

To show $s \vee t \in S \cup_{\gamma} T$, we assume again that $S \cup_{\gamma} T \subseteq \{r\}^{\lhd}$, which is equivalent to $S \cup T \subseteq \{r\}^{\lhd}$, again using that $R^{\lhd\rhd\lhd} = R^{\lhd}$, for all $R \subseteq Tm$, and that γ is a closure operator. So, we have $s, t \in \{r\}^{\lhd}$, namely $s \; N \; r$ and $t \; N \; r$. Using $(L\vee)$ we obtain $s \vee t \; N \; r$; thus $s \vee t \subseteq \{r\}^{\lhd}$.

Consider the assignment $x \mapsto \{x\}^{\lhd}$ on all variables, and extend it to a homomorphism $t \mapsto \bar{t}$ from $\mathbf{Tm}$ to $\mathbf{Tm}^{+}$.

Lemma 2. For every term r, we have $r \in \bar{r} \subseteq \{r\}^{\lhd}$.

Proof. When r is a variable, the statement follows by (ref). We proceed by induction on the structure of r. If $r = s \wedge t$ and the lemma holds for s and t, then by using Lemma 1 we have $s \wedge t \in \bar{s} \cap \bar{t} \subseteq \{s \wedge t\}^{\lhd}$; note that $\overline{s \wedge t} = \bar{s} \cap \bar{t}$. Likewise $s \vee t \in \overline{s \vee t} \subseteq \{s \vee t\}^{\lhd}$, as $\overline{s \vee t} = \bar{s} \cup_{\gamma} \bar{t}$.

Transitivity elimination

Lemma 2. For every term r, we have $r \in \bar{r} \subseteq \{r\}^{\lhd}$.

Theorem. An equation is valid in lattices iff it is derivable in Lat^{-} iff it is derivable in Lat.

Proof. Clearly every equation that is derivabe in Lat^{-} is also derivable in Lat, and every equation derivable in Lat is valid in lattice-ordered posets and thus also in lattices.

Now assume that the equation $s \leqslant t$ is valid in lattices. Then it also holds in the lattice $\mathbf{Tm}^{+}$ under the valuation $r \mapsto \bar{r}$, namely $\bar{s} \subseteq \bar{t}$. Using Lemma 2, we obtain that $s \in \bar{s} \subseteq \bar{t} \subseteq \{t\}^{\lhd}$; hence $s \; N \; t$. Therefore, $s \leqslant t$ is derivable in Lat^{-}.

Corollary. The equational theory of lattices is decidable.

$\mathbf{Tm}^{+}$ is the Dedekind-MacNeille completion of the free lattice.

Whitman's conditions: $s \wedge t \leqslant r \vee q$ holds in lattices iff one of the following holds in lattices:

$$s \wedge t \leqslant r, \quad s \wedge t \leqslant q, \quad s \leqslant r \vee q, \quad t \leqslant r \vee q.$$

Lattice frames

A *lattice frame* is a triple $\mathbf{F} = (A, B, N)$, where A and B are sets and $N \subseteq A \times B$. We denote by $\mathbf{F}^+$ the associated complete lattice on $\gamma_N[\mathcal{P}(A)]$.

A *Gentzen lattice frame* is a pair $(\mathbf{F}, \mathbf{S})$, where $\mathbf{F} = (A, B, N)$ is a lattice frame, S is a common subset of A and B, $\mathbf{S} = (S, \wedge, \vee)$ is a *partial* algebra and the following implications hold:

$$\frac{s_1 \; N \; b \quad s_2 \; N \; b}{s_1 \vee s_2 \; N \; b} \qquad \frac{a \; N \; s_1}{a \; N \; s_1 \vee s_2} \qquad \frac{a \; N \; s_2}{a \; N \; s_1 \vee s_2}$$

$$\frac{a \; N \; s_1 \quad a \; N \; s_2}{a \; N \; s_1 \wedge s_2} \qquad \frac{s_1 \; N \; b}{s_1 \wedge s_2 \; N \; b} \qquad \frac{s_2 \; N \; b}{s_1 \wedge s_2 \; N \; b}$$

as well as $s \; N \; s$, for all $a \in A$, $b \in B$, $s, s_1, s_2 \in S$, provided the operations in the denominators are defined. A Gentzen lattice frame is called *transitive*, if it also satisfies:

$$\frac{a \; N \; s \quad s \; N \; b}{a \; N \; b}$$

for all $a \in A$, $b \in B$, $s \in S$. It is called *antisymmetric* if for all $s, s' \in S$:

$$\frac{s \; N \; s' \quad s' \; N \; s}{s = s'}$$

Lattice frames

Lemma. Let $(\mathbf{S}, \mathbf{F})$ be a Gentzen lattice frame. Then for $s, t \in S$ and $S, T \in F^+$, if $s \in S \subseteq \{s\}^{\lhd}$, $t \in T \subseteq \{T\}^{\lhd}$, then $s \wedge t \in S \cap T \subseteq \{s \wedge t\}^{\lhd}$, $s \vee t \in S \cup_\gamma T \subseteq \{s \vee t\}^{\lhd}$. We will say that $q : \mathbf{S} \to \mathbf{F}^+$, $q(s) = \{s\}^{\lhd}$ is a *quasi-homomorphism*.

Lemma. If $(\mathbf{S}, \mathbf{F})$ is a transitive Gentzen lattice frame, then q is a homomorphism (of partial algebras). If it is also antisymmetric, then q is an embedding.

Example. If $\mathbf{P}$ is a poset, we consider the frame $\mathbf{F_P} = (P, P, \leqslant)$. Also, we define the partial algebra $\mathbf{S} := (P, \wedge, \vee)$, where $a \vee b$ is the least common upper bound of a and b in the poset $\mathbf{P}$ and $a \wedge b$ is the greatest common lower bound, if they exist.

Then $(\mathbf{S}, \mathbf{F_P})$ is a transitive and antisymmetric Gentzen lattice frame, $q(s) = \{s\}^{\lhd} = {\downarrow} s$. Also, $\mathbf{F_P^+}$ is the Dedekind-MacNeille completion of $\mathbf{P}$. By the Lemma we get that $q : \mathbf{S} \to \mathbf{F_P^+}$ is an embedding of partial lattices, hence it is an order-preserving map that preserves existing joins and meets.

Note: For a finite lattice $\mathbf{L}$, instead of $(L, L, \leqslant)$ we can take $(\mathcal{J}(\mathbf{L}), \mathcal{M}(\mathbf{L}), \leqslant)$ to get a lattice frame, and the Galois algebra will also be $\mathbf{L}$.

Note: For a finite distributive lattice $\mathbf{L}$, we do not need two sorts: the poset $(\mathcal{J}(\mathbf{L}), \leqslant)$ suffices (Priestley, Kripke). Then take downets. Or take two sorts $(\mathcal{J}(\mathbf{L}), \mathcal{J}(\mathbf{L}), \not\geqslant)$ and do Galois.

Lattice expansions

A *Boolean algebra* $\mathbf{A} = (A, \wedge, \vee, \neg, 1, 0)$ is a bounded distributive complemented ($x \wedge \neg x = 0$ and $x \vee \neg x = 1$) lattice. Alternatively:

A *Boolean algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \neg, 1, 0)$ such that

- $(A, \wedge, \vee, 0, 1)$ is a bounded ~~distributive~~ lattice,
- for all $a, b, c \in A$, $a \wedge b \leqslant c \Leftrightarrow b \leqslant \neg a \vee c$. (residuation)
- $\neg\neg x = x$ (involutivity).

Standard examples: Powersets. Algebraic semantics for classical propositional logic.

A *lattice-ordered group* $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ is a group and a lattice such that multiplication is order preserving/distr. over $\wedge$/distr. over $\vee$. Alternatively:

An *ℓ-group* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid
- for all $a, b, c \in A$, $a \cdot b \leqslant c \Leftrightarrow b \leqslant a^{-1} \cdot c \Leftrightarrow a \leqslant c \cdot b^{-1}$. (residuation)
- $(x^{-1})^{-1} = x$ (involutivity).

Standard examples: Order-preserving permutations over a chain.

Algebras of relations

Let X be a set and $Rel(X) = \mathcal{P}(X \times X)$ be the set of all binary relations on X. For the structure $(Rel(X), \cap, \cup, ^c, \varnothing, X^2, \cdot, +, \sim)$, we have that

- $(Rel(X), \cap, \cup, \varnothing, X^2)$ is a bounded lattice
- for all $R, S, T \in Rel(X)$, $R \cap S \subseteq T \iff S \subseteq R^c \cup T \iff R \subseteq T \cup S^c$.
- $R^{cc} = R$.
- $(Rel(X), \cdot, 1)$ is a monoid
- for all $R, S, T \in Rel(X)$, $R \cdot S \subseteq T \iff S \subseteq {\sim}R + T \iff R \subseteq T + {\sim}S$.
- ${\sim}{\sim}R = R$.

For relations R, and S, we denote by

- $R \cdot S$ the *relational* composition of R and S
- 1 is the equality/diagonal relation on X
- R^c the complement and by $R^\cup$ the converse of R and set ${\sim}R = R^{c\cup} = R^{\cup c}$.
- $R + S = {\sim}({\sim}S \cdot {\sim}R)$ the DeMorgan dual of relational composition

Three examples

Boolean algebra: $\mathbf{A} = (A, \wedge, \vee, \top, \neg)$

- $(A, \wedge, \vee)$ is a lattice, $(A, \wedge, \top)$ is a monoid
- $a \wedge b \leqslant c \Leftrightarrow b \leqslant \neg a \vee c$
- $\neg\neg a = a$.

lattice-ordered group: $\mathbf{L} = (L, \wedge, \vee, \cdot, ^{-1}, 1)$

- $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, 1)$ is a monoid
- $a \cdot b \leqslant c \Leftrightarrow b \leqslant a^{-1} \cdot c \Leftrightarrow a \leqslant c \cdot b^{-1}$.
- $(a^{-1})^{-1} = 1$.

Relation algebra: $(L, \wedge, \vee, \top, \neg, \cdot, 1, +, \sim)$

- $(L, \wedge, \vee)$ is a lattice, $(L, \wedge, \top)$ is a monoid, $(L, \cdot, 1)$ is a monoid,
- $a \wedge b \leqslant c \Leftrightarrow b \leqslant \neg a \vee c$
- $a \cdot b \leqslant c \Leftrightarrow b \leqslant {\sim}a + c \Leftrightarrow a \leqslant c + {\sim}b$.
- $\neg\neg a = a$ and ${\sim}{\sim}a = a$.
- $\neg(x + y) = \neg x \cdot \neg y$ (the other DeMorgan-type equations follow: ${\sim}(x \vee y) = {\sim}x \wedge {\sim}y$, $\neg(x \vee y) = \neg x \wedge \neg y$, ${\sim}(x + y) = {\sim}y \cdot {\sim}x$).

Heyting algebras

In a Boolean algebra we define: $a \to b = \neg a \vee b$.

Actually, Boolean algebras are equivalent to structures $\mathbf{A} = (A, \wedge, \vee, \to, \bot, \top)$ such that (we define $\neg a = a \to \bot$)

- $(A, \wedge, \vee, \bot, \top)$ is a bounded lattice,
- for all $a, b, c \in A$, $a \wedge b \leqslant c \Leftrightarrow b \leqslant a \to c$ ($\wedge$-residuation)
- for all $a \in A$, $\neg\neg a = a$ (alt. $a \vee \neg a = \top$).

A Heyting algebra is a structure $\mathbf{A} = (A, \wedge, \vee, \to, \bot, \top)$ such that

- $(A, \wedge, \vee, \bot, \top)$ is a bounded lattice,
- for all $a, b, c \in A$, $a \wedge b \leqslant c \Leftrightarrow b \leqslant a \to c$ ($\wedge$-residuation)

Examples: Given a topological space (X, τ), $(\tau, \cap, \cup, \to, \varnothing, X)$ is a Heyting algebra, where $a \to b = (\neg a \vee b)^o$, the topological interior.

Heyting algebras algebraic semantics for intuitionistic propositional logic. They form the basis for constructive mathematics and parts of computer science (BHK interpretation).

Residuated lattices

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a monoid and
- for all $a, b, c \in A$, $ab \leqslant c \Leftrightarrow b \leqslant a\backslash c \Leftrightarrow a \leqslant c/b$ (residuation)

Residuated lattices form an equational class RL, hence a variety (closed under H, S, P).

A residuatd lattice is called *commutative* if its monoid is commutative (BA, HA), *distributive* if its lattice is distributive (BA, HA, RA, LG), *integral* if it satisfies $x \leqslant 1$ (BA, HA). In the commutative case $x\backslash y = y/x$ and we write $x \to y$ for the common value.

We often expand the signature with an arbitrary constant 0 (*pointed residuated lattices*), which allows us to define negation(s): $\neg x := x \to 0$; also $x^r := x\backslash 0$ and $x^\ell := 0/x$.

In ℓ-groups we have $0 = 1$, $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$. Also, LG = RL + $(xx^r = 1)$.

HA = pRL + $(xy = x \wedge y)$.

Further examples of residuated lattices

If $\mathbf{M} = (M, \cdot, e)$ is a monoid, for $X, Y \subseteq M$ we define:

$$X \cdot Y = \{x \cdot y : x \in X,\ y \in Y\}$$

$$X \backslash Y = \{z \in M : X \cdot \{z\} \subseteq Y\}, \qquad Y/X = \{z \in M : \{z\} \cdot X \subseteq Y\}.$$

Fact. $(\mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{e\})$ is a residuated lattice.

Let $\mathbf{R}$ be a ring with unit and let $\mathcal{I}(\mathbf{R})$ be the set of all (two-sided) ideals of $\mathbf{R}$.

Then $(\mathcal{I}(\mathbf{R}), \cap, \vee)$ is a lattice, where $I \vee J = \{i + j : i \in I, j \in J\}$. For $I, J \in \mathcal{I}(\mathbf{R})$:

$$IJ = \{\sum_{fin} ij : i \in I, j \in J\}, \quad I\backslash J = \{k : Ik \subseteq J\}, \quad J/I = \{k : kI \subseteq J\}.$$

Fact. [Ward and Dilworth 1930's] $(\mathcal{I}(\mathbf{R}), \cap, \vee, \cdot, \backslash, /, R)$ is a residuated lattice.

Further examples of residuated lattices:

1. Boolean algebras, ℓ-groups, Relation algebras, Heyting algebras.
2. Locales/frames in point-free topology $(L, \wedge, \bigvee)$ (def. equiv. to complete HAs).
3. Quantales (relating to C*-algebras, quantal-valued model theory, physics).
4. Computer Science: Action algebras, Kleene algebras with tests. (Pratt, Kozen)
5. Also, MV-algebras and other algebras of substructural logics:
   Linear, relevance, MV, BL, MTL, where multiplication is strong conjunction.

Substructural logics

Classical logic studies truth.

Intuitionistic logic (Brouwer, Heyting) deals with provability or constructibility.
The algebraic models are Heyting algebras.

Many-valued logic (Łukasiewicz) allows different degrees of truth. [Ulam's game]
$[x \to (x \to y)] \to (x \to y)$ is not a theorem. The algebraic models fail $x \leqslant x \cdot x$.

Relevance logic (Anderson, Belnap) deals with relevance.
$p \to (q \to q)$ is not a theorem. The algebraic models do not satisfy integrality $x \leqslant 1$.
$p \to (\neg p \to q)$ [or $(p \cdot \neg p) \to q$] is not a theorem, where $\neg p = p \to 0$. The algebraic models do not satisfy $0 \leqslant x$.

Linear logic (Girard) studies preservation of resourses.
$p \to (p \to p)$ [or $(p \cdot p) \to p$] and $p \to (p \cdot p)$ are not theorems.
The algebraic models do not satisfy mingle $x^2 \leqslant x$ nor contraction $x \leqslant x^2$.

Related deductive systems appear in:

- Mathematical linguistics: Context-free grammars, pregroups. (Lambek, Buzskowski)
- CS: Memory allocation, pointer management, concurrent programming. (Separation logic, bunched implication logic).

Gentzen's system **LJ** for intuitionistic logic

A ***sequent*** is an expression $a_1, \dots, a_n \Rightarrow a_0$, where a's are formulas. For $a, b, c \in Fm$, $x, y, z, x_1, x_2 \in Fm^*$, we have the inference rules:

$$\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c}\ (\text{cut}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id})$$

$$\frac{y, x_1, x_2, z \Rightarrow c}{y, x_2, x_1, z \Rightarrow c}\ (\text{e}) \qquad \frac{y, z \Rightarrow c}{y, x, z \Rightarrow c}\ (\text{w}) \qquad \frac{y, x, x, z \Rightarrow c}{y, x, z \Rightarrow c}\ (\text{c})$$

$$\frac{y, a, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \text{L}\ell) \quad \frac{y, b, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \text{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \text{R})$$

$$\frac{y, a, z \Rightarrow c \quad y, b, z \Rightarrow c}{y, a \vee b, z \Rightarrow c}\ (\vee \text{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \text{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \text{R}r)$$

$$\frac{x \Rightarrow a \quad y, b, z \Rightarrow c}{y, x, a \to b, z \Rightarrow c}\ (\to \text{L}) \qquad \frac{a, x \Rightarrow b}{x \Rightarrow a \to b}\ (\to \text{R})$$

$$\frac{y, z \Rightarrow c}{y, 1, z \Rightarrow c}\ (1\text{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\text{R})$$

Basic substructural logics

In $\mathbf{LJ}$, the sequent $a_1, \ldots, a_n \Rightarrow a_0$ is provable iff the sequent $a_1 \wedge \ldots \wedge a_n \Rightarrow a_0$ is, so comma corresponds to $\wedge$. The proof system $\mathbf{FL}$ of Full Lambek calculus is obtained from Gentzen's proof system $\mathbf{LJ}$ for intuitionistic logic by removing the three basic structural rules:

$$\frac{u[x,y] \Rightarrow c}{u[y,x] \Rightarrow c}\ (e) \qquad \text{(exchange)} \qquad [x \to (y \to z)] \to [y \to (x \to z)] \qquad xy \leqslant yx$$

$$\frac{u[x,x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \text{(contraction)} \qquad [x \to (x \to y)] \to (x \to y) \qquad x \leqslant x^2$$

$$\frac{u[\varepsilon] \Rightarrow c}{u[x] \Rightarrow c}\ (i) \qquad \text{(integrality)} \qquad y \to (x \to y) \qquad x \leqslant 1$$

In $\mathbf{FL}$, comma and $\wedge$ do not correspond any more. But we can conservatively add a new connective $\cdot$ (*fusion* or *multiplication*) that does correspond to comma and rules:

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c}\ (\cdot \mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b}\ (\cdot \mathrm{R})$$

Also, $a \to b$ splits into $a \backslash b$ and b / a.

FL

$$\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c}\ (\text{cut}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id})$$

$$\frac{y, a, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \mathsf{L}\ell) \quad \frac{y, b, z \Rightarrow c}{y, a \wedge b, z \Rightarrow c}\ (\wedge \mathsf{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \mathsf{R})$$

$$\frac{y, a, z \Rightarrow c \quad y, b, z \Rightarrow c}{y, a \vee b, z \Rightarrow c}\ (\vee \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}r)$$

$$\frac{x \Rightarrow a \quad y, b, z \Rightarrow c}{y, x, (a \backslash b), z \Rightarrow c}\ (\backslash \mathsf{L}) \qquad \frac{a, x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \mathsf{R})$$

$$\frac{x \Rightarrow a \quad y, b, z \Rightarrow c}{y, (b/a), x, z \Rightarrow c}\ (/\mathsf{L}) \qquad \frac{x, a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathsf{R})$$

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c}\ (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b}\ (\cdot \mathsf{R})$$

$$\frac{y, z \Rightarrow c}{y, 1, z \Rightarrow c}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathsf{R})$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$. Extensions of $\mathbf{FL}$ are known as *substructural logics*; for example $\mathbf{FL_c}$, $\mathbf{FL_{ec}}$ etc. Varieties of residuated lattices form *equivalent algebraic semantics* for various substructural logics (a la Lindenbaum-Blok-Pigozzi) [G.-Ono 2006].

Residuated frames

A *residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, \varepsilon)$ where W and W' are sets $N \subseteq W \times W'$, $(W, \circ, \varepsilon)$ is a monoid and for all $x, y \in W$ and $w \in W'$ there exist subsets $x \mathbin{\backslash\!\!\backslash} w, w \mathbin{/\!\!/} y \subseteq W'$ such that

$$(x \circ y)\ N\ w \ \Leftrightarrow\ y\ N\ (x \mathbin{\backslash\!\!\backslash} w) \ \Leftrightarrow\ x\ N\ (w \mathbin{/\!\!/} y)$$

Notation $X^{\triangleright} := \{w' \in W' : X\ N\ w'\}$, $Y^{\triangleleft} := \{w \in W : w\ N\ Y\}$, $\gamma(X) := X^{\triangleright\triangleleft}$.

The *Galois algebra* $\mathbf{W}^{+} = \gamma[\mathcal{P}(W, \circ, \varepsilon)]$ is a residuated lattice. $[\cap, \backslash, /, \ \cup_{\gamma}, \cdot_{\gamma}, \gamma(1)]$

<u>Day 2: Undecidability</u>

- Via encoding semigroups
- Via Counter Machines
- Via Projective Geometry
- Via Tag systems
- Via Wang tilings

Encoding semigroups

Observation [2022 Jipsen-Tsinakis]. The quasiequational theory of residuated lattices is undecidable.

Recall that a *quasiequation* is the universal closure of an implication:

$$s_1 = t_1 \ \& \ \dots \ \& \ s_n = t_n \Longrightarrow s = t,$$

where n could be 0. A class of algebras is axiomatized by a set of quasiequations iff it is a *quasivariety*: closed under $\mathsf{ISPP_U}$.

Proof. Let $\mathcal{S}$ be the class of semigroup subreducts of residuated lattices (up to isomorphism). Given a semigroup $\mathbf{S}$, we can add a unit to obtain a monoid $\mathbf{S}_1$, and we can consider the residuated lattice $\mathcal{P}(\mathbf{S}_1)$; since the singletons of $\mathcal{P}(\mathbf{S}_1)$ form a semigroup isomorphic to $\mathbf{S}_1$, we have $\mathbf{S} \in \mathcal{S}$. So, $\mathcal{S}$ is the class of all semigroups. So, a semigroup quasiequation holds in $\mathcal{S}$ iff it holds in residuated lattices.
If the quasiequational theory of residuated lattices were decidable, then we could decide equations in semigroups, but this problem is known to be undecidable.

Theorem. If a variety contains all powersets of monoids, then it has an undecidable quasi-equational theory. For example, the variety of distributive residuated lattices.

Unfortunately, the same argument breaks down in the commutative case.

For example, Heyting algebras have a decidable quasiequational theory. (Even the FEP).

Counter machines: hardware

Counter machines store numbers and can *increment*, *decrement* or *test* if the number is zero. The hardware of a *counter machine* consists of

- a finite set $R = \{r_1, \ldots, r_k\}$ of *registers* (or counters), which can be thought of as empty boxes labeled by the name of the register, and *tokens* each of which can be in some register,
- a final set Q of internal *states* in which the machine can be in, with designated initial state q_I and final state q_F.

A *configuration* consists of a state and a natural number for each register. The configuration of a machine can be represented by the monoid term

$$q \mid r_1^{n_1} \mid r_2^{n_2} \mid \cdots \mid r_k^{n_k} \mid \text{ or } qS_0 r_1^{n_1} S_1 r_2^{n_2} S_2 \cdots S_{k-1} r_k^{n_k} S_k.$$

The auxiliary letters from $S = \{S_0, \ldots, S_k\}$ are called *stoppers*. The machine will be able to move, via applications of instructions, to other configurations during the computation:

$$qS_0 r_1^{n_1} S_1 r_2^{n_2} \cdots S_{k-1} r_k^{n_k} S_k \leqslant q' S_0 r_1^{m_1} S_1 r_2^{m_2} \cdots S_{k-1} r_k^{m_k} S_k \leqslant \ldots \leqslant q_F S_0 \ldots S_k$$

and will be *accepted* if there is some way to reach $q_F S_0 \cdots S_k$ (non-determinism).

Counter machines: software

The software consists of a finite set P of instructions taken from three different types in addition to all instructions $qx \leqslant xq$ and $xq \leqslant qx$, where $q \in Q$, and $x \in R \cup S$.

- Increment instructions $qS_i \leqslant q'r_iS_i$: when in state q, increment register r_i by one token and change the internal state to q'.

  Intention: $qS_0r_1^{n_1} \cdots S_{i-1}r_i^{n_i}S_i \cdots r_k^{n_k}S_k \leqslant q'S_0r_1^{n_1} \cdots S_{i-1}r_i^{n_i+1}S_i \cdots r_k^{n_k}S_k$.

- Decrement instructions $qr_iS_i \leqslant q'S_i$: when in state q, decrement register r_i (if possible) by one token and change the internal state to q'.

  Intention: $qS_0r_1^{n_1} \cdots S_{i-1}r_i^{n_i+1}S_i \cdots r_k^{n_k}S_k \leqslant q'S_0r_1^{n_1} \cdots S_{i-1}r_i^{n_i}S_i \cdots r_k^{n_k}S_k$.

- Zero-test instructions $S_{i-1}qS_i \leqslant S_{i-1}q'S_i$: when in state q, check the contents of register r_i and if they are empty then move to state q'.

  Intention: $qS_0r_1^{n_1} \cdots S_{i-1}r_i^{0}S_i \cdots r_k^{n_k}S_k \leqslant q'S_0r_1^{n_1} \cdots S_{i-1}r_i^{0}S_i \cdots r_k^{n_k}S_k$.

The *computation relation* $\leqslant$ of a machine is defined as the reflexive-transitive closure of the smallest compatible (with multiplication) relation containing the instructions.

We say that a configuration C is *accepted* if $C \leqslant q_FS_0S_1 \cdots S_k$.

Counter machines: example

For example, consider the machine that has set of states $Q = \{q_1, q_F\}$, with initial and final state q_F, set of registers $R = \{r_1, r_2\}$ and set of instructions $P = \{q_F r_1 S_1 \leqslant q_1 S_1,\ q_1 r_2 S_2 \leqslant q_F S_2\}$, then we have

$$\begin{aligned} q_F S_0 r_1 S_1 r_2 S_2 \quad & \leqslant S_0 q_F r_1 S_1 r_2 S_2 \\ & \leqslant S_0 q_1 S_1 r_2 S_2 \\ & \leqslant S_0 S_1 q_1 r_2 S_2 \\ & \leqslant S_0 S_1 q_F S_2 \\ & \leqslant S_0 q_F S_1 S_2 \\ & \leqslant q_F S_0 S_1 S_2. \end{aligned}$$

The only initial configurations that are accepted are of the form $q_F S_0 r_1^n S_1 r_2^n S_2$, where n is a natural number, so the machine checks if we have an equal number of r_1-tokens as r_2-tokens.

Counter machines undecidability

It is well known that there is a 2-counter machine with an undecidable set of accepted configurations. All computations in the machine are interpreted as valid in the residuated lattice presented by relations corresponding to the instructions. So RL satisfies the quasiequation

$$\&P \;\Rightarrow\; u \leqslant q_F S_0 S_1 \cdots S_k$$

where P is the set of instructions of an undecidable machine and u is an accepted configuration of the machine.

Conversely, if some configuration is not accepted then we can construct a residuated lattice that falsifies the quasiequation; it will be the Galois algebra of a residuated frame.

Let M be a machine and $W := (Q \cup R_k \cup S)^*$ be the free monoid generated by $Q \cup R_k \cup S$ and $W' = W \times W$. We define the relation $N \subseteq W \times W'$ via

$$x\ N\ (u, v) \quad \text{iff} \quad uxv \leqslant q_F S_0 S_1 \cdots S_k,$$

for all $x, z \in W$. Observe that, for any $x, y, u, v \in W$,

$$xy\ N\ (u, v) \iff uxyv \leqslant q_F S_0 S_1 \cdots S_k \iff x\ N\ (u, yv) \iff y\ N\ (ux, v).$$

Theorem. $\mathbf{W} := (W, W', N, \circ, \varepsilon)$ is a residuated frame, $\mathbf{W}^+ \in$ RL, and there exists a valuation $\nu : \mathbf{Fm} \to \mathbf{W}^+$ that falsifies the quasiequation of the machine.

Counter machines undecidability

Corollary. Let M be a 2-counter machine with an undecidable set of accepted configurations and let P be its the set of instructions. Then, a configuration u is accepted in M iff RL satisfies the quasiequation

$$\&P \;\Rightarrow\; u \leqslant q_F S_0 S_1 \cdots S_k.$$

Corollary. The word problem for residuated lattices is undecidable; i.e., there is a finitely-presented residuated lattice $\langle X|P\rangle$ for which it is undecidable whether a pair of terms (s,t) correspond to the same element of the residuated lattice.

For the subvariety axiomatized by the equation $x^2y^2 = y^2x^2$, the computation relation needs to allow for such transitions of the form $x^2y^2 \leqslant y^2x^2$. As these do not correspond to an instruction of the machine, they may be viewed as a *glitch* or a *bug* in the computation which may be applied at any time without our control.

Fortunately, even though instances of $x^2y^2 \leqslant y^2x^2$ are available, they cannot be applied to any configuration in a non-trivial way, due to the position of the stoppers. (*Resilience*)

Corollary. The word problem for the variety of residuated lattices axiomatized by $x^2y^2 = y^2x^2$ is undecidable. Same for every variety containing $\mathbf{W}^+$.

Unfortunately, the encoding does not work for contraction $x \leqslant x^2$. Nor for commutativity $xy = yx$: tokens can move past the stoppers and then the zero-test may not be implemented correctly. A different encoding is needed.

Handling contraction

The encoding will be resilient to (unaffected by) the presence of the contraction glitch $x \leqslant x^2$, if every configuration avoids containing instances of squares of monoid words, i.e., if every configuration is ***square-free***.

So, we should modify the form $q_i S_0 a^n S_1 a^m S_2$ of configurations of a 2-CM.

Fact. Given the 3-letter alphabet $\{a, b, c\}$, the monoid homomorphism $h : \{a, b, c\}^* \to \{a, b, c\}^*$ extending

$$h(a) = abc,\ h(b) = ac, \text{ and } h(c) = b,$$

preserves square-freeness. In particular, $a, h(a), \ldots, h^n(a)$, etc, are all square-free words.

The (mirror image of h) function $g : \{a, b, c\}^* \to \{a, b, c\}^*$, defined by $g(x) = \overline{h(\overline{x})}$ where $\overline{w}$ denotes the mirror image of the word w, also preserves square-freeness.

We change the representation of configurations into the form $q_i S_0 g^n(a) S_1 h^m(a) S_2$, or even better:

$$A g^n(a) B q_i C h^m(a) D$$

where $\{A, B, C, D\}$ is a set of stoppers.

Handling contraction

The instruction that increments the first register and moves from q_i to q_j

$$\begin{aligned} Ag^n(a)Bq_iCh^m(a)D &\leqslant Ad_1\cdots d_kB^+q_jCh^m(a)D \\ &\leqslant Ad_1\cdots B^+g(d_k)q_jCh^m(a)D \\ &\leqslant AB^+g(d_1)\cdots g(d_k)q_jCh^m(a)D \\ &\leqslant ABe_1\cdots e_\ell q_jCh^m(a)D \\ &\leqslant Ae_1\cdots e_\ell Bq_jCh^m(a)D \\ &= Ag^{n+1}(a)Bq_jCh^m(a)D \end{aligned}$$

is obtained by a sequence of reductions (and auxiliary stoppers B^+, B^-, C^+, C^-):

$$Bq_i \leqslant B^+q_j \qquad dB^+ \leqslant B^+g(d) \qquad AB^+ \leqslant AB \qquad Bd \leqslant dB$$

where $d_1, \ldots, d_k, e_1, \ldots, e_\ell \in \{a, b, c\}$ be such that:

$g^n(a) =: d_1\cdots d_k$ and $g^{n+1}(a) = g(g^n(a)) = g(d_1\cdots d_k) = g(d_1)\cdots g(d_k) =: e_1\cdots e_\ell$.

Theorem. [Horcik, JPAA 2015] The word problem for $\mathbf{FL_c}$ is undecidable. The same holds all varieties containing the one axiomatized by $(x \leqslant x^2) + (x^3 \leqslant x^2)$.

Theorem. The same holds for hereditarily-square equations such as $xy \leqslant x^2y \vee xy^2$.

Handling commutativity

The quasiequational theory in the $\{\cdot, 1, \leqslant\}$-fragment of commutative residuated lattices is actually decidable, so no encoding will work in this language.

To get an encoding that works we involve the connective of join $\vee$ to implement *parallel computation*. The main strand of the computation proceeds on transitioning to the next state by the zero-test without any restrictions, while the auxiliary computation safeguards that the zero-test is applied correctly, by terminating only when the value of the register was empty.

Therefore, we consider joins of configurations, which we call *instantaneous descriptions*, IDs. and we represent by

$$C_1 \vee \cdots \vee C_m,$$

where the C_i's are configurations; so ID's of the machine are elements of the free join-semilattice over the set QR^*. We assume that this sits inside a commutative idempotent semiring generated by $Q \cup R$. Due to commutativity, stoppers play no role (and are omitted) and monoid words are of the form

$$qr_1^{n_1} r_2^{n_2} \cdots r_k^{n_k}.$$

We call this formalization an *And-branching Counter Machine*.

And-branching Counter machines: software

The software consists of a finite set P of instructions taken from three different types.

- Increment instructions $q \leqslant q' r_i$: when in state q, increment register r_i by one token and change the internal state to q'.
  Intended application: $q r_1^{n_1} r_2^{n_2} \cdots r_i^{n_i} \cdots r_k^{n_k} \leqslant q' r_1^{n_1} r_2^{n_2} \cdots r_i^{n_i+1} \cdots r_k^{n_k}$.
- Decrement instructions $q r_i \leqslant q'$: when in state q, decrement register r_i (if possible) by one token and change the internal state to q'.
  Intended application: $q r_1^{n_1} r_2^{n_2} \cdots r_i^{n_i+1} \cdots r_k^{n_k} \leqslant q' r_1^{n_1} r_2^{n_2} \cdots r_i^{n_i} \cdots r_k^{n_k}$.
- OMIT: Zero-test instructions $q \leqslant q'$: when in state q, check the contents of register r_i and if they are empty then move to state q'.
  Intended application: $q r_1^{n_1} r_2^{n_2} \cdots r_i^{0} \cdots r_k^{n_k} \leqslant q' r_1^{n_1} r_2^{n_2} \cdots r_i^{0} \cdots r_k^{n_k}$.
- Copy instructions $q \leqslant q' \vee q''$: when in state q, duplicate the data and move to states q' and q''.
  Intended application: $q r_1^{n_1} \cdots r_k^{n_k} \leqslant q' r_1^{n_1} \cdots r_k^{n_k} \vee q'' r_1^{n_1} \cdots r_k^{n_k}$.

This works well as in RLs: $qR \leqslant (q' \vee q'')R = q'R \vee q''R$.

The *computation relation* $\leqslant$ of a machine is defined as the reflexive-transitive closure of the smallest compatible (with multiplication and join) relation containing the instructions.

Different encoding

Recall that in lattices

$$C_1 \vee \cdots \vee C_m \leqslant q_F \;\Leftrightarrow\; (C_1 \leqslant q_F \;\&\; \ldots \;\&\; C_m \leqslant q_F)$$

so $\vee$ behaves conjunctively: all parallel computations/branches must be accepted.

Example: Let M_{even} be a machine with only one counter r three states q_0, q_1, q_F (where $q_I = q_0$), and three instructions which we even name for reference as p_1, p_2, p_3:

$$q_0 r \leqslant^{p_1} q_1 \qquad q_1 r \leqslant^{p_2} q_0 \qquad q_0 \leqslant^{p_3} q_F$$

Note that $q_0 r^n \leqslant q_F$ iff n is even. For example:

$$q_0 r^4 \leqslant^{p_1} q_1 r^3 \leqslant^{p_2} q_0 r^2 \leqslant^{p_1} q_1 r \leqslant^{p_2} q_0 \leqslant^{p_3} q_F$$

$$q_0 r^3 \leqslant^{p_1} q_1 r^2 \leqslant^{p_2} q_0 r \leqslant^{p_3} q_F r$$

Examples, the zero test

Fact. ACMs can simulate CMs.

Zero-test (for r_3): Let $R = \{r_1, r_2, r_3\}$, $Q = \{q_I, q', z_3, q_F\}$ and assume that the instructions include

$$q_I \leqslant q' \vee z_3 \qquad z_3 r_1 \leqslant z_3 \qquad z_3 r_2 \leqslant z_3 \qquad z_3 \leqslant q_F$$

and possibly other instructions of the form $q' \leqslant \ldots$.

Notice that

$$q_I r_1^2 r_2 r_3 \leqslant (q' \vee z_3) r_1^2 r_2 r_3 = q' r_1^2 r_2 r_3 \vee z_3 r_1^2 r_2 r_3 \leqslant \ldots q' r_1^2 r_2 r_3 \vee q_F r_3$$

Since $q_F r_3 \nleqslant q_F$ the process does not terminate.

However,

$$q_I r_1^2 r_2 \leqslant (q' \vee z_3) r_1^2 r_2 = q' r_1^2 r_2 \vee z_3 r_1^2 r_2 \leqslant \ldots \leqslant q_F \vee q_F = q_F$$

if we assume that due to other instructions we get $q' r_1^2 r_2 \leqslant q_F$

So, these instructions block termination if there are r_3 tokens, but allow the continuation of the passage to q' (in the context of a non-harmful $\_ \vee q_F$) if the contents of r_3 are empty.

Theorem [Lincoln et al., 1992] The quasiequational theory of CRL is undecidabile.

Commutative extensions

We cannot combine the ideas for $x \leqslant x^2$ and $xy = yx$ (for example M_{even} accepts all IDs). Actually, we will see that $\mathbf{FL_{ec}}$ is decidable.

Note that the inequality $x \leqslant x^2 \vee x^4$ plus $xy = yx$ also causes problems with the encoding: *it can lead to termination when it is not intended*:

Let $\leqslant$ be the computation relation of M_{even} and let $\leqslant'$ be the one where we also add the ambient inequality $x \leqslant x^2 \vee x^4$.

On one hand we have $q_0 r^3 \nleqslant q_F$ since 3 is odd. On the other hand, $q_0 r^3 \leqslant' q_F$, because

$$q_0 r^3 = q_0 r^2 r \leqslant' q_0 r^2 r^2 \vee q_0 r^2 r^4 = q_0 r^4 \vee q_0 r^6 \leqslant' q_F,$$

since $q_0 r^4 \leqslant q_F$ and $q_0 r^6 \leqslant q_F$.

However, this time we will show that unlike with $x \leqslant x^2$ we can get undecidability, by lifting the encoding to the exponents.

Idea

We consider $x \leqslant x^2 \vee x^4$ applied to r^t, $t \neq 0$:

$$qr^n = qr^s r^t \leqslant' qr^s(r^{2t} \vee r^{4t}) = qr^{s+2t} \vee qr^{s+4t}$$

However, not both $s + 2t$ and $s + 4t$ can be powers of K, if $K \geqslant 3$.

Idea: Declare that we expect the contents of all registers to be powers of K and that instructions (or at least blocks of instruction applications) always stay with powers of K. Then no application of $x \leqslant x^2 \vee x^4$ can help with termination.

To achieve that we implement the undecidable 2-counter machine M inside a machine M_K where contents n of a register in M are replaced with K^n in M_K. The instructions of M are replaced by *programs* in M_K.

Conditions for existence of K

We focus on *simple* equations: $t_0 \leqslant t_1 \vee \cdots \vee t_n$, where the t_i's are products of variables and t_0 has no repetitions. E.g., $xy \leqslant xy^2 \vee x^2y$.

A simple equation is called *prespinal* if there is a substitution that yields an inequality of the form

$$x_1 \cdots x_n \leqslant 1 \vee x_1^{k_{11}} \vee x_1^{k_{12}} x_2^{k_{22}} \vee \cdots \vee x_1^{k_{1n}} \cdots x_n^{k_{nn}}$$

where $k_{ii} \neq 0$, and where the term $1\vee$ could be missing.

Theorem [G.-St.John, JSL 2022] The word problem is undecidable for every variety axiomatized by a simple non-prespinal equation.

Examples: $x \leqslant x^2 \vee x^3$,
$xy \leqslant x \vee x^2y \vee y^2$,
$xyzw \leqslant x^2yzw \vee x^3y^2z^2w^2$.

The machine M_K

Let M be the 2-register undecidable machine; we describe the new machine M_K. It has one more counter r_3. It contains all the states of M, it has three additional states z_1, z_2, z_3 in order to implement zero-tests and also contains states that are internal to its subprograms below. All copy instructions remain as they are, but we replace the add-one and substract-one instructions by multiply-by-K and divide-by-K programs.

$$\begin{array}{ccccc} q & \leqslant & q'r & \Longrightarrow & qr^{\forall} \sqsubseteq q'r^{K\cdot\forall} \\ qr & \leqslant & q' & \Longrightarrow & qr^{\forall} \sqsubseteq q'r^{K\backslash\forall} \end{array}$$

We obtain, for each $q \in Q$,

$$qr_1^{n_1}r_2^{n_2} \leqslant_M q_f \iff qr_1^{K^{n_1}}r_2^{K^{n_2}} \leqslant_{M_K} q_F.$$

Programs

The **add-K program**: $q \sqsubseteq q' r_i^K$, for $i \in \{1,2\}$, adds K tokens to register r_i, with input state q and output state q'.

$$q \leqslant a_1 r_i, \quad a_1 \leqslant a_2 r_j, \quad \cdots, \quad a_{K-1} \leqslant q' r_i.$$

The **transfer program**: $t_0 r_i^{\forall} \sqsubseteq q' r_j^{\forall}$, transfers all contents of register r_i to register r_j, with input state q and output state q'.

$$t_0 r_i \leqslant t_1, \qquad t_1 \leqslant t_0 r_j, \qquad t_0 r_i^{\varnothing} \sqsubseteq q'.$$

The **multiply by** K program: $q r_i^{\forall} \sqsubseteq q' r_i^{K \cdot \forall}$, for each $i \in \{1,2\}$, multiplies the contents of r_i by K, with input state q and output state q'.

$$q r_3^{\varnothing} \sqsubseteq c, \quad c r_i \leqslant a_0, \quad a_0 \sqsubseteq c r_3^K, \quad c r_i^{\varnothing} \sqsubseteq t_0, \quad t_0 r_3^{\forall} \sqsubseteq q' r_i^{\forall}.$$

The **end program**: $q_f r_1 r_2 \sqsubseteq q_F$ transitions from the final state q_f of M to the final state q_F of M_K.

$$q_f r_1 \leqslant c_F, \quad c_F r_2 \leqslant q_F.$$

Proof idea

The inequality/glitch can modify the rules in a linear way. However, the contents in the register are stored in an exponential way.

We can prove that after an application of a non-prespinal inequality if the resulting registers in all configurations/joinands have powers of K, then the original contents of the registers were one of these configurations. This proves that applications of the inequality are actually redundant; the machine is *resilient* to applications of the inequality.

That this works for all non-prespinal inequalities uses positive linear algebra:

Moving to additive notation, the values of the exponents in the join in the RHS can be organized in a matrix R. The substitution that exhibits possible prespinality can be written by a matrix S. The spinal form corresponds to an *upper triangular matrix* T.

The problem reduces to whether there exist S and upper triangualar T, both with natural number coefficients, such that $SR = T$. By rearranging rows (relabeling variables) and columns (simultaneously in S and R), both S and R become block-upper-triangular.

Prespinality of R is proved to be equivalent to the existence of positive solutions to systems of inequalities given by lower-right submatrices of R.

We prove that there exists a large enough value of K exhibiting the admissibility of non-prespinal rules. This is done by moving from $\mathbb{N}$ to $\mathbb{R}$ and using the theorem of alternatives of positive linear algebra.

2-tag systems

A *2-tag system* consists of a finite alphabet A and a function $u : A \to A^*$. A word $a_1 \cdots a_n \in A^*$, with $n \geqslant 2$, is transformed into $a_3 \cdots a_n u(a_1)$, i.e., the first two letters are deleted and a new word (depending on the first letter) is appended in the end. The system *terminates* on a given input if it is transformed to a word of length at most 1.

Fact. It is undecidable whether a given 2-tag system terminates on a given input.

Theorem [Chvalovsky JSL 2015]. The quasiequational theory for non-associative residuated lattices is undecidable.

Proof. We encode each 2-tag system (A, u). If A is the alphabet and $u_i := u(a_i)$, we prove that the 2-tag system terminates on w iff

$$\&P \implies e\overline{w}X \leqslant eX \vee eS$$

is provable in non-associative residuated lattices (association to the right), where $e, e', X, X', S, c_i^j, C_i, C_i', D, D', d, d'$ are new symbols, for $a_1, \ldots, a_n \in A$,

$\overline{w}$ is defined by $\overline{a_1 \cdots a_n} = c_1^2 \ldots c_{n-1}^n$, and $\overline{a_1 \cdots a_{n+1}} = c_1^2 \ldots c_{n-1}^n c_{n+1}$,

for n: even, and P includes the following list of inequalities:

$$c_j^k X \leqslant c_j^k \overline{u_i} X' \vee c_j^k C_i \qquad c_j X \leqslant \overline{a_j u_i} X' \vee c_j C_i$$

$$ec_i X \leqslant eX \qquad ee' S \leqslant eS$$

2-tag systems

$$c_jX \leqslant \overline{a_j u_i}X' \vee c_jC_i \qquad c_j^kX \leqslant c_j^k\overline{u_i}X' \vee c_j^kC_i \qquad c_j^kc_\ell C_i \leqslant c_j^kC_i \qquad ec_i^jC_i \leqslant eS$$

$$c_i^j \leqslant e' \vee d' \quad ee'S \leqslant eS \quad c_j^kc_i^\ell D' \leqslant c_j^kD' \quad X' \leqslant D' \quad ed'D' \leqslant eS \quad e'eX \leqslant e'X$$

For example, for the implementation of the 2-tag system on $A = \{a_1, a_2\}$ with $u(a_1) = a_2$ and $u(a_2) = a_1$, we have the following computation for input $a_1a_2a_2$:

$$\begin{aligned}
ec_1^2c_2X &\leqslant ec_1^2(c_2^2X' \vee c_2C_1) = ec_1^2c_2^2X' \vee ec_1^2c_2C_1\\
&\leqslant e(e' \vee d')c_2^2X' \vee ec_1^2C_1\\
&\leqslant ee'c_2^2X' \vee ed'c_2^2X' \vee eS\\
&\leqslant ee'(c_2^2c_1X \vee c_2^2C_2') \vee ed'c_2^2D' \vee eS\\
&\leqslant ee'c_2^2c_1X \vee ee'c_2^2C_2' \vee ed'D' \vee eS\\
&\leqslant ee'(e \vee d)c_1X \vee ee'S \vee eS \vee eS\\
&\leqslant ee'ec_1X \vee ee'dc_1X \vee eS \vee eS \vee eS \vee eS\\
&\leqslant ee'eX \vee ee'dc_1D \vee eS \leqslant ee'X' \vee ee'dD \vee eS\\
&\leqslant eX \vee ee'D \vee eS \leqslant eX \vee ee'X \vee eS \leqslant eX \vee eX \vee eS
\end{aligned}$$

A mismatch $e \cdots X'$ or $e' \cdots X$ indicates that we have appended and we need to delete.

Projective geometry (for CDRL)

A modular n-frame in a lattice $\mathbf{L}$ is an $n \times n$ matrix, $C = [c_{ij}]$, $c_{ij} \in L$, such that: (for $a_i = c_{ii}$ and $e = \bigwedge\{a_i \mid i \in \mathbb{N}_n\}$; $\mathbb{N}_n = \{1, ..., n\}$),

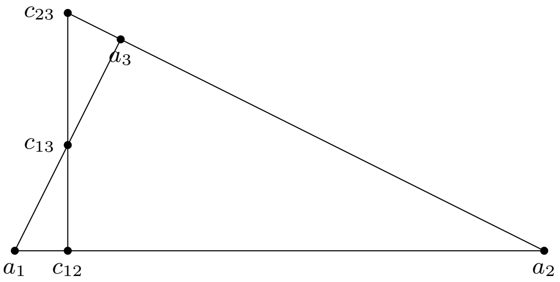
i) $\bigvee A_1 \wedge \bigvee A_2 = \bigvee(A_1 \cap A_2)$, for all $A_1, A_2 \subseteq \{a_1, a_2, ..., a_n\}$, where $\bigvee \varnothing = e$;

ii) $c_{ij} = c_{ji}$, for all $i, j \in \mathbb{N}_n$;

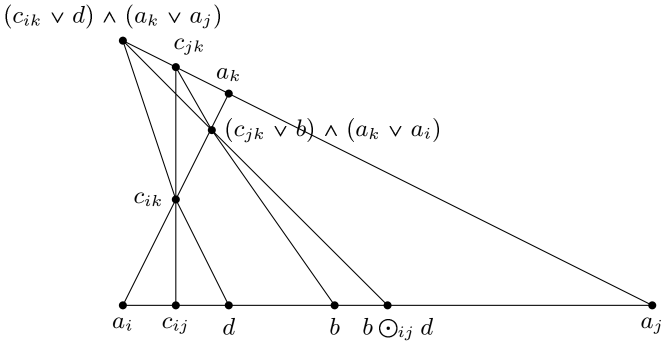
iii) $a_i \vee a_j = a_i \vee c_{ij}$, for all $i, j \in \mathbb{N}_n$;

iv) $a_i \wedge c_{ij} = e$, for all distinct $i, j \in \mathbb{N}_n$;

v) $(c_{ij} \vee c_{jk}) \wedge (a_i \vee a_k) = c_{ik}$, for all distinct triples $i, j, k \in \mathbb{N}_n$.



Multiplication and addition



If $[c_{ij}]$ is a modular n-frame in a modular lattice $\mathbf{L}$, we define

- $L_{ij} = \{x \in \mathbf{L} \mid x \vee a_j = a_i \vee a_j \text{ and } x \wedge a_j = e\}$, for all distinct $i, j \in \mathbb{N}_n$;
- $b \otimes_{ijk} d = (b \vee d) \wedge (a_i \vee a_k)$, for all $b \in L_{ij}, d \in L_{jk}$;
- $b \odot_{ij} d = (b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d)$, for all $b, d \in L_{ij}$;
- $b \oplus_{ij} d = [((b \vee c_{ik}) \wedge (a_j \vee a_k)) \vee ((d \vee a_k) \wedge (a_j \vee c_{ik}))] \wedge (a_i \vee a_j)$, $b, d \in L_{ij}$.

Fact. [Lipshitz] The definitions of $\odot_{ij}$ and $\oplus_{ij}$ are independent of the choice of $k \in \mathbb{N}_n$, for $i \neq k \neq j$.

The associated ring in modular lattices

Theorem [Von Neumann 1960] Let $C = [c_{ij}]$ be a modular n-frame in a modular lattice $\mathbf{L}$, where $n \geqslant 4$. Then $\mathbf{R}_{ij} = (L_{ij}, \oplus_{ij}, \odot_{ij}, a_i, c_{ij})$ is a ring for all distinct $i, j \in \mathbb{N}_n$. Moreover, all rings $\mathbf{R}_{ij}$ are isomorphic and are called the *ring associated with the modular n-frame C of $\mathbf{L}$*.

For a vector space, $\mathbf{V}$, denote by $L(\mathbf{V})$ the set of all subspaces of $\mathbf{V}$. It is well known that $\mathbf{L}(\mathbf{V}) = (L(\mathbf{V}), \wedge, \vee)$ is a modular lattice, where meet is intersection and the join of two subspaces is the subspace generated by their union.

Lemma [Lipshitz] Let $\mathbf{V}$ be an infinite-dimensional vector space. Then,

i) $\mathbf{L}(\mathbf{V})$ contains a 4-frame, C, where e is the least element of $\mathbf{L}(\mathbf{V})$ and

ii) Any countable semigroup is a subsemigroup of the multiplicative semigroup of the ring associated with C.

Theorem [Lipshitz TAMS 1974] The word problem for modular lattices is undecidable.

The residuated case

A *residuated-lattice n-frame* (or just *n-frame*) in a residuated lattice $\mathbf{L}$ is an $n \times n$ matrix, $C = [c_{ij}]$, $c_{ij} \in \mathbf{L}$, (set $a_i = c_{ii}$), such that:

i) $a_i a_j = a_j a_i$, for all $i, j \in \mathbb{N}_n$;

ii) $\prod A_1 \wedge \prod A_2 = \prod (A_1 \cap A_2)$, for all $A_1, A_2 \subseteq \{a_1, a_2, ..., a_n\}$, where $\prod \varnothing = 1$;

iii) $a_i^2 = a_i$, for all $i \in \mathbb{N}_n$;

iv) $c_{ij} c_{jk} \wedge a_i a_k = c_{ik}$, for all distinct triples $i, j, k \in \mathbb{N}_n$;

v) $c_{ij} = c_{ji}$, for all $i, j \in \mathbb{N}_n$;

vi) $c_{ij} a_j = a_i a_j$, for all $i, j \in \mathbb{N}_n$;

vii) $c_{ij} \wedge a_j = 1$, for all distinct $i, j \in \mathbb{N}_n$.

An element a of a residuated lattice $\mathbf{L}$ is called *modular* if $c(b \wedge a) = cb \wedge a$ and $(a \wedge b)c = a \wedge bc$ for all elements b, c of L, such that $c \leqslant a$. An n-frame of a residuated lattice is called *modular* if $\prod A$ is modular, for all $A \subseteq \{a_1, a_2, ..., a_n\}$.

If $[c_{ij}]$ is an n-frame in a residuated lattice $\mathbf{L}$, we define

- $L_{ij} = \{x \in \mathbf{L} \mid x a_j = a_i a_j \text{ and } x \wedge a_j = 1\}$, for all distinct $i, j \in \mathbb{N}_n$;
- $b \otimes_{ijk} d = bd \wedge a_i a_k$, for all $b \in L_{ij}, d \in L_{jk}$;
- $b \odot_{ij} d = (b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d)$, for all $b, d \in L_{ij}$ and for all distinct i, j, k.

The associated semigroup

Theorem. Let $C = [c_{ij}]$ be a modular residuated-lattice 4-frame in a residuated lattice $\mathbf{L}$. Then, $\mathbf{S}_{12} = (L_{12}, \odot_{12})$ is a semigroup, called *the semigroup associated with the 4-frame* C.

Lemma. Let $\mathbf{L}$ be a distributive residuated lattice, with a top element, T, and a bottom element, B. If $a, \tilde{a}, \in L$, $a^2 \leqslant a$, $a\tilde{a} \leqslant \tilde{a}$, $\tilde{a}a \leqslant \tilde{a}$, $a \wedge \tilde{a} = B$ and $a \vee \tilde{a} = T$, then, a is modular.

Let $\mathbf{V}$ be a vector space and let A, B be elements of the power set $\mathcal{P}(V)$ of $\mathbf{V}$. We define $1 = \{0_{\mathbf{V}}\}$ and

$$A \wedge B = A \cap B, A \vee B = A \cup B, AB = \{a+b | a \in A, y \in B\}, A\backslash B = B/A = \{c | \{c\}A \subseteq B\}.$$

Recall that $\mathcal{P}(V) = (\mathcal{P}(V), \wedge, \vee, \cdot, \backslash, /, 1)$ is a distributive residuated lattice.

Moreover, $L(\mathbf{V})$ is a subset of $\mathcal{P}(V)$, but $\mathbf{L}(\mathbf{V})$ is not a sublattice of (the lattice reduct of) $\mathcal{P}(V)$.

Nevertheless, $\wedge_{\mathbf{L}(\mathbf{V})} = \wedge_{\mathcal{P}(V)}$ and $\vee_{\mathbf{L}(\mathbf{V})} = \cdot_{\mathcal{P}(V)}$.

Undecidability of quasiequations for CDRL

If $\mathbf{S} = (S, \bullet)$, $S = (x_1, x_2, ..., x_n \mid r_1^\bullet(\overline{x}) = s_1^\bullet(\overline{x}), ..., r_k^\bullet(\overline{x}) = s_k^\bullet(\overline{x}))$, is a finitely presented semigroup and $\mathcal{V}$ is a variety of residuated lattices, let $\mathbf{L}(\mathbf{S}, \mathcal{V})$ be the residuated lattice in $\mathcal{V}$ with the presentation described below: (we define $\mathcal{A}(C) = \mathcal{P}(\{a_1, a_2, a_3, a_4\})$)

Generators: $x'_1, x'_2, ..., x'_n$, c_{ij} $(i, j \in \mathbb{N}_4)$, $\top, \bot$ and $\widetilde{\prod A}$ $(A \in \mathcal{A}(C))$. Relations:

i) Equations (i)-(vii) of the definition (for $n = 4$);
ii) $x'_i a_2 = a_1 a_2$ and $x'_1 \wedge a_2 = 1$, for all $i \in \mathbb{N}_n$;
iii) $r_i^{\odot_{12}}(\overline{x}') = s_i^{\odot_{12}}(\overline{x}')$, for all $i \in \mathbb{N}_k$, where $t^{\odot_{12}}$ denotes the evaluation of t in the semigroup associated with the 4-frame $[c_{ij}]$;
iv) $\bot^2 = \bot$, $\top^2 = \top = \top/\bot = \bot/\bot = \bot\backslash\top = \bot\backslash\bot$, $\bot \leqslant 1 \leqslant \top$ and $\bot \leqslant x \leqslant \top$, $\bot x = x\bot = \bot$, for every generator x;
v) $x^2 \leqslant x$, $x\tilde{x} \leqslant \tilde{x}$, $\tilde{x}x \leqslant \tilde{x}$, $x \wedge \tilde{x} = \bot$ and $x \vee \tilde{x} = \top$, for all x of the form $\prod A, A \in \mathcal{A}(C)$.

Let $R(\overline{x})$ denote the conjunction $\bigwedge_{i \in \mathbb{N}_k} r_i(\overline{x}) = s_i(\overline{x})$ of the relations of $\mathbf{S}$ and $R'(\overline{x}', C, \overline{\mathcal{A}}(C), \bot, \top)$ the conjunction of the relations of $\mathbf{L}(\mathbf{S}, \mathcal{V})$; $\overline{x}$ abbreviates $(x_1, x_2, ..., x_n)$.

Theorem. [G. 2002] If $\mathcal{V}$ is a variety such that $\mathsf{HSP}(\mathcal{P}(V)) \subseteq \mathcal{V} \subseteq \mathsf{DRL}$, for some infinite-dimensional vector space $\mathbf{V}$, then $\mathcal{V}$ has an undecidable quasi-equational theory. E.g., every variety in the interval $[\mathsf{CDRL}, \mathsf{DRL}]$.

Wang tilings

A *(Wang) tile* is a 4-tuple $t = (t_1, t_2, t_3, t_4) \in \mathbb{N}^4$; we think of t as a square tile with '*colors*' t_1, t_2, t_3, t_4 on its up, down, left, and right edge, respectively, and we define $\mathsf{U}(t) = t_1, \mathsf{D}(t) = t_2, \mathsf{L}(t) = t_3, \mathsf{R}(t) = t_4$.

Given a finite set of tiles $\mathcal{W}$, we say that $\mathcal{W}$ *tiles* $\mathbb{N}^2$, if there is a function $\tau : \mathbb{N}^2 \to \mathcal{W}$ such that adjacent tiles have matching colors on their common sides, i.e., for all $(m, n) \in \mathbb{N}^2$,

$$U(\tau(m,n)) = D(\tau(m,n+1)) \text{ and } R(\tau(m,n)) = L(\tau(m+1,n)).$$

In this case, we say that τ is a *tiling (of $\mathbb{N}^2$ with $\mathcal{W}$)*.

Fact. It is undecidable whether a finite set of Wang tiles can tile $\mathbb{N}^2$.

The algebraic semantics the relevance logic $\mathbf{R}$ is the variety of distributive commutative residuated lattices with contraction $x \leqslant x^2$; so it is in the interval [CDRL, DRL] and has undecidable quasi-equational theory. The logic $\mathbf{S}$ is the $\{\wedge, \vee \to\}$ fragment of $\mathbf{R}$; the algebraic semantics is the class of subreducts of the models of $\mathbf{R}$.

Theorem [Knudstorp 2024]. The equational theory of the logic $\mathbf{S}$ is undecidable.

Proof idea. Due to distributivity, the relational semantics are one-sorted: a poset (corresponding to the poset of join irreducibles in the finite case) and a ternary relation to capture $\to$. The encoding works on these relational semantics.

Wang tilings

Let

$$\psi_1 = [e \wedge (e \to e)] \to o, \qquad \psi_2 = [o \wedge (o \to o)] \to e$$
$$\psi_3 = (e \vee o) \to \{(e \vee o) \wedge [(e \vee o) \to (e \vee o)]\}$$
$$\psi = (o \wedge \psi_1 \wedge \psi_2 \wedge \psi_3) \to e$$

Claim. ψ is refuted only by infinite models. (In particular, $\mathbf{S}$ does not have the finite model property.)

In such an (infinite) model there is an infinite chain $z_0 < z_1 < z_2 < \dots$. Such a chain can be used to define an axis (the x-axis and the y-axis).

Claim. For every set of tiles $\mathcal{W}$ there is a formula $\psi_{\mathcal{W}}$ (that does not fit in one slide) such that: $\mathcal{W}$ tiles $\mathcal{N}^2$ iff $\psi_{\mathcal{W}}$ fails in $\mathbf{S}$.

The proof uses the implication connective to transfer the failure from one point of the poset to a next one and utilizing the axis to create a grid. Then to points of the grid we assign colors based on what formulas are valid there.

Day 3: Decidability and complexity

- Decidability via residuated frames
- Decidability via diagrams
- Upper bounds via Proof theory
- Lower bounds via Ack-bounded CMs

The FEP for n-potent

A class of algebras $\mathcal{K}$ has the *finite embeddability property* if for every $\mathbf{A} \in \mathcal{K}$ and finite $B \subseteq A$, there is a finite algebra $\mathbf{D} \in \mathcal{K}$ such that B embeds in $\mathbf{D}$ as a partial algebra.

If a universal formula fails in $\mathcal{K}$, then it fails in a finite algebra of $\mathcal{K}$. So, if a universal class is finitely axiomatized and has the FEP then its universal theory is decidable.

The proof for Boolean algebras is easy: Take $\mathbf{D}$ to be the subalgebra of $\mathbf{A}$ generated by B; it is a finite subalgebra of $\mathbf{A}$ containing B.

Heyting algebras are not locally finite. Take $\mathbf{C}$ to be the $\{\vee, \wedge, 1\}$-subalgebra of $\mathbf{A}$ generated by B; it is a finite $\{\vee, \wedge, 1\}$-subalgebra of $\mathbf{A}$ containing B (distributive lattices are locally finite). Also, since $\mathbf{C}$ is a finite distributive lattice it supports a (unique) Heyting algebra expansion $\mathbf{D}$. We define $x \to y := \bigvee\{z \in C \mid x \wedge z \leqslant y\}$.

If $\mathbf{A}$ is a commutative residuated lattices that satisfies $x^n = x^{n+1}$ (*n-potency*), we take $\mathbf{C}$ to be the $\{\vee, \cdot, 1\}$-subalgebra of $\mathbf{A}$ generated by $B = \{b_1, \ldots, b_k\}$, then the elements of $\mathbf{C}$ are joins of products $b_1^{n_1} \cdots b_k^{n_k}$, where $0 \leqslant n_i \leqslant n$ for all i. So, $\mathbf{C}$ is finite. Also, because $\mathbf{C}$ supports a residuated lattice expansion $\mathbf{D}$: $x \to y := \bigvee\{z \in C \mid x \cdot z \leqslant y\}$.

But what if we have a (weaker) axiom of the form $x^n \leqslant x^m$ (*knotted inequality*)?

Fact. The quasiequation $1 \leqslant x$ and $xy = 1 \Longrightarrow x = 1$ holds in every finite residuated lattice, but fails in the ℓ-group $\mathbb{Z}$. So, the FEP fails in every variety of residuated lattices containing $\mathbb{Z}$. For example, the FEP fails in RL and CRL.

Residuated frames

A *residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, \varepsilon)$ where W and W' are sets $N \subseteq W \times W'$, $(W, \circ, \varepsilon)$ is a monoid and for all $x, y \in W$ and $w \in W'$ there exist subsets $x \backslash\!\!\backslash w, w /\!\!/ y \subseteq W'$ such that

$$(x \circ y)\ N\ w \ \Leftrightarrow\ y\ N\ (x \backslash\!\!\backslash w) \ \Leftrightarrow\ x\ N\ (w /\!\!/ y)$$

Notation $X^{\triangleright} := \{w' \in W' : X\ N\ w'\}$, $Y^{\triangleleft} := \{w \in W : w\ N\ Y\}$, $\gamma(X) := X^{\triangleright\triangleleft}$.

The *Galois algebra* $\mathbf{W}^{+} = \gamma[\mathcal{P}(W, \circ, \varepsilon)]$ is a residuated lattice. $[\cap, \backslash, /, \cup_\gamma, \cdot_\gamma, \gamma(1)]$

Example 1: If $\mathbf{L}$ is a RL, $\mathbf{W_L} = (L, L, \leqslant, \cdot, \{1\})$ is a residuated frame. Moreover, for $\mathbf{W_L}$, $x \mapsto \{x\}^{\triangleleft}$ is an embedding. The underlying poset of $\mathbf{W}^{+}_{\mathbf{L}}$ is the *Dedekind-MacNeille completion* of $\mathbf{L}$.

Example 2: We define the frame $\mathbf{W_{FL}}$, where (sequent: $a_1, \ldots, a_n \Rightarrow a_0$.)

- $(W, \circ, \varepsilon)$ is the free monoid over the set Fm of all formulas
- $W' = S_W \times Fm$, where S_W is the set of all *contexts* $u[x] = y{\circ}x{\circ}z$ of W,
- $x\ N\ (u, a)$ iff $\vdash_{\mathbf{FL}} u[x] \Rightarrow a$.
- $(u, a) /\!\!/ x = \{(u[\_ \circ x], a)\}$ and $x \backslash\!\!\backslash (u, a) = \{(u[x \circ \_], a)\}$

$$\begin{aligned} x \circ y N (u, a) \quad & \text{iff } \vdash_{\mathbf{FL}} u[x \circ y] \Rightarrow a \\ & \text{iff } \vdash_{\mathbf{FL}} u[x{\circ}y] \Rightarrow a \\ & \text{iff } xN(u[\_ \circ y], a) \qquad \text{iff } yN(u[x \circ \_], a). \end{aligned}$$

Gentzen frames

A residuated frame $\mathbf{W} = (W, W', N, \circ, \varepsilon)$ together with a common subset S of W and W' that is a partial algebra $\mathbf{S}$, is called a *Gentzen frame* if it satisfies versions of the rules of $\mathbf{FL}$ such as: for $a, b \in S$, $z \in W'$ with $a \wedge b$ defined in S,

$$\frac{aNz \quad bNz}{a \vee bNz} \ (\vee\mathsf{L})$$

[G.-Jipsen, TAMS, 2013] The map $s \mapsto s^{\lhd}$ is a partial quasihomomorphism of $\mathbf{S}$ to $\mathbf{W}^+$.

A Gentzen system has the *finite model property* (FMP), if for every sequent that is not provable there exists a finite countermodel; the variety is generated by its finite members.

For a sequent s, $s^{\leftarrow}$ is the set of all sequents involved in an exhaustive proof search of s:

- $s \in s^{\leftarrow}$
- if $(\{t_1, t_2\}, t)$ is an instance of a rule of $\mathbf{L}$ and $t \in s^{\leftarrow}$, then $t_1, t_2 \in s^{\leftarrow}$.

Theorem. $\mathbf{FL}$ has the FMP. (Also, all simple extensions that do not increase complexity.)

Given a sequent s, not provable in $\mathbf{FL}$, we modify the residuated frame $\mathbf{W}_{\mathbf{FL}}$ to get $\mathbf{W}_{\mathbf{FL},s}$: Let N' be the relation defined by

$$x \ N' \ (u, a) \qquad \text{iff} \qquad x \ N \ (u, a) \ \text{ or } \ (u(x) \ \Rightarrow \ a) \notin s^{\leftarrow}.$$

We can prove that $\mathbf{W}_{\mathbf{FL},s}$ is a residuated Gentzen frame, that it is finite, and that s fails in $\mathbf{W}^+_{\mathbf{FL},s}$.

Knotted rules

$$\frac{u[x,x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \frac{u[x] \Rightarrow c}{u[x,x] \Rightarrow c}\ (m) \qquad \frac{u[x,x,x] \Rightarrow c}{u[x,x] \Rightarrow c}\ (k(2,3))$$

The above three rules correspond to the algebraic axioms $x \leqslant x^2$, $x^2 \leqslant x$ and $x^2 \leqslant x^3$. In general *knotted rules* $k(n,m)$ allow for controlled duplication of resources and correspond to $x^n \leqslant x^m$. For $m > n$: *knotted contraction rule*; for $m < n$: *knotted weakening*.

Unfortunately, adding (m) or (k(3,2)) breaks cut-elimination for the calculus. However, we can get an equivalent structural rule that preserves cut elimination, the *linearization* of $x^2 \leqslant x^3$: $xy \leqslant x^3 \vee x^2y \vee xy^2 \vee y^3$.

$$\frac{u[x,x,x] \Rightarrow c \quad u[x,x,y] \Rightarrow c \quad u[x,y,y] \Rightarrow c \quad u[y,y,y] \Rightarrow c}{u[x,y] \Rightarrow c}\ (k(2,3)')$$

Fact. Every equation over $\{\vee, \cdot, 1\}$ is equivalent to a conjunction of simple equations: $t_0 \leqslant t_1 \vee \cdots \vee t_n$, where t_i are $\{\cdot, 1\}$-terms and t_0 is linear.

Theorem [G.-Jipsen, TAMS, 2013]: The structural rule corresponding to the simple equation $\varepsilon : t_0 \leqslant t_1 \vee \cdots \vee t_n$ preserves cut elimination

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a}\ (R(\varepsilon))$$

Also, any $\{\vee, \cdot, 1\}$ equations that hold in $\mathbf{W}$ are preserved in $\mathbf{W}^+$.

FEP via residuated frames

A class of algebras $\mathcal{K}$ has the *finite embeddability property* if for every $\mathbf{A} \in \mathcal{K}$ and finite $B \subseteq A$, there is a finite algebra $\mathbf{D} \in \mathcal{K}$ such that B embeds in $\mathbf{D}$ as a partial algebra.

For the FEP application we consider the frame $\mathbf{W_{A,B}}$, where

- $\mathbf{C} = (W, \cdot, 1)$ is the submonoid of $\mathbf{A}$ generated by B,
- $W' = S_B \times B$, where S_W is the set of all *unary linear polynomials* $u[x] = y \circ x \circ z$ of $(W, \cdot, 1)$,
- $x \; N \; (u, b)$ iff $u[x] \leqslant_{\mathbf{A}} b$,
- $(u, a) \mathbin{/\!\!/} x = \{(u[\_ \cdot x], a)\}$ and $x \mathbin{\backslash\!\!\backslash} (u, a) = \{(u[x \cdot \_], a)\}$.

Then the Galois algebra $\mathbf{D} := \mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is a residuated lattice that satisfies all $\{\vee, \cdot, 1\}$ equations that hold in $\mathbf{A}$. (So the FEP holds also for such axiomatic extensions.) The map $b \mapsto \{(id, b)\}^{\lhd}$ is an embedding of the partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ into $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$.

To prove that $\mathbf{D}$ is finite, we use that $\mathbf{C}$ is a wqo for the 1-generated part, and then commutativity to extend to the $|B|$-generated part. Also, we can relax commutativity to $xyx = xxy$ or to $xyxzx = yx^3zy$ or to $xyxzxwx = x^2yzxx^2$, etc. (weak commutativity).

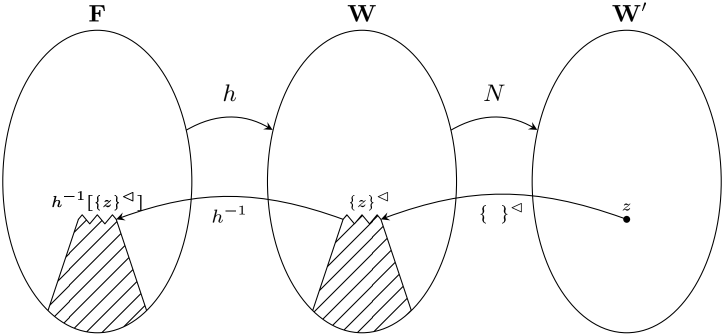
Theorem [van Alten, JSL 2005; G.-Cardona, IJAC, 2015] The algebra $\mathbf{D}$ is finite.

(Note that knotted rules are prespinal, so they avoid the undecidability obstacle of counter machines. Also, some commutativity is needed as $\mathbf{FL_c}$ is undecidable.)

The construction

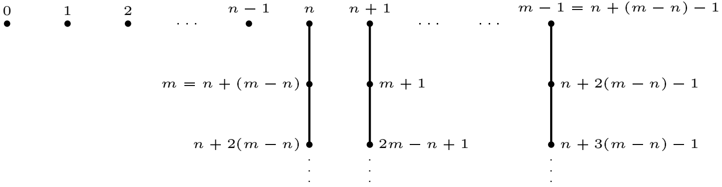
To prove that $\mathbf{D}$ is finite, it suffices to prove that there are only finitely many basic closed sets.

We considering the free pomonoid $\mathbf{F}$ based on the knotted rule and the weak commutativity, and a order preserving subjective homomorphism $h : \mathbf{F} \to \mathbf{W}$.



The 1-generated case, commutativity and weak commutativity

The 1-generated free pomonoid for $x^m \leqslant x^n$ is $(\mathbf{N}, +, 0, \leqslant_n^m)$, where $u \leqslant_n^m v$ if and only if $u = v$, or $n \leqslant v < u$ and $u \equiv v \pmod{m-n}$.



The order can be controlled, as the pomonoid is a *dually well ordered*. This can be used to show finiteness of the 1-generated part.

With commutativity, in the finitely-generated case we get a single form $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$. So, the infinite behavior is moved to the 1-generated case, where it is tamed by using wqos.

With weak commutativity, potentially infinitely many form can appear, such as $x_1^{n_1} x_2^{n_2} x_3^{n_3}$, $x_2^{n_2} x_1^{n_1} x_3^{n_3}$, $x_2^{n_2} x_1^{n_1} x_2^{n_2'} x_3^{n_3} x_1^{n_1'}$. We study the dynamics of the Zimin-like words and prove that we get finitely-many forms.

Known complexities

We write $\mathbf{FL_{ec}}$ for $\mathbf{FL} + (e) + (c)$, etc, whre $c : x \leqslant x^2$, $e : xy = yx$, and $w : x \leqslant 1$.

Provability of intuitionistic logic $\mathbf{FL_{ecw}}$ (the *equational theory* of the variety of Heyting algebras) is PSPACE-complete. [Statman, TCS, 1979]
The same holds for $\mathbf{FL}$, $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$. [Horčik-Terui, TCS, 2011]

However, provability of $\mathbf{FL_{ec}}$ is Ackermanian-complete [Urquhart, JSL, 1999].

Provability of $\mathbf{FL_c}$ is undecidable [Chvalovský-Horčik, JSL, 2016].

Deducibility (*quasiequational theory*) for $\mathbf{FL_{ew}}$ is TOWER-complete [Tanaka, CSL, 2023], for $\mathbf{FL_{ec}}$ is Ackermanian complete [Urquhart, JSL, 1999], for $\mathbf{FL_e}$ is undecidable [Lincoln et all, APAL, 1992].

Deducubility for most structural extensions of $\mathbf{FL_e}$ (varieties of commutative residuated lattices) are undecidable [G.-St.John, JSL, 2022].

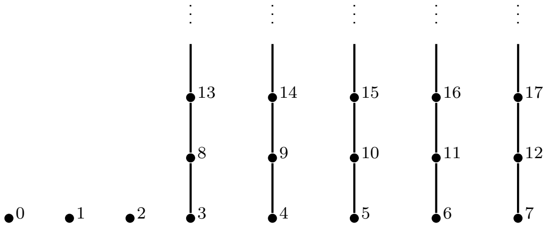
Note that if the logic (e.g., $\mathbf{FL_{ec}}$) has a deduction theorem $\Gamma, \phi \vdash \psi \Leftrightarrow \Gamma \vdash \phi \to \psi$, then provabilty (equational) and deducibility coincide (quasiequational theory).

These special cases generalize to a uniform class of logics/varieties. The generalization occurs in multiple directions.

Well-ordered sets

In a poset antichains and infinite descending chains are special cases of bad sequences. In a quasi-ordered set, a sequence $a_1, a_2, \ldots$ is called *bad* if for all $i < j$ we have $a_i \not\leqslant a_j$. A qosets that lack bad sequencs is called a well-quasi-ordered set (a wqo).
A bad sequence in $(\mathbb{N}, \leqslant)$: $9, 8, 5, 2, 1$. $(\mathbb{N}, \leqslant)$ is a wqo.
A bad sequence in $(\mathbb{N}^2, \leqslant)$: $(2,3), (2,2), (100,1), (99,1), (50,1), (2,1)$.
Some key features of wqos:

- Products and disjoint unions of wqo's are wqo's.
- Every finitely-generated downset is finite. (Mathematical induction.)
- Every upset is finitely generated.



So, the monoid $\mathbf{C}$ generated by B is infinite but also a wqo $\mathbb{N}^k_{n,m}$ (in the figure is $\mathbb{N}_{3,8}$ for the case $x^3 \leqslant x^8$).

Proof-theoretic analysis for upper bounds

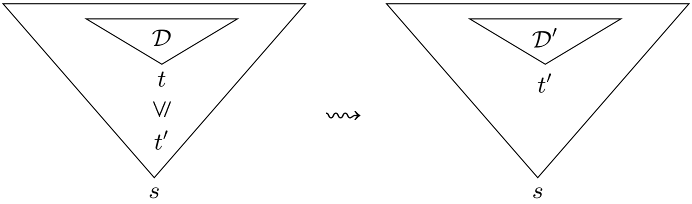
We will start by following (Kripke, JSL, 1959): provability of $\mathbf{FL_{ec}}$ is decidable.
Given a sequent s we want to check if it is provable in $\mathbf{FL_{ekR}}$, where k is a knotted contraction rule and R is any finite set of simple rules.

We call Ω the set of all formulas in s and consider only Ω-sequents in our analysis. We fix a listing of Ω so, modulo commutativity, every Ω-sequent is an element of $\mathbb{N}^{\Omega}$.

We design a new cut-free calculus $\mathbf{FL^*_{ekR}}$ that is equivalent to $\mathbf{FL_{ekR}}$ and

- the logical rules include a fixed number, $g(k, R)$, of applications of k below them.
- does not contain k but does contain its linearization k'. (This differs from Kripke.)

We prove 'Curry's Lemma' for $\mathbf{FL^*_{ekR}}$: If t has a derivation $\mathcal{D}$ of height at most h and $t' \leqslant t$ (in the wqo $\mathbb{N}^{\Omega}_k$), then t is also has a derivation $\mathcal{D}'$ of height at most h.



Proof-theoretic analysis for upper bounds

So if a proof has a branch where t' appears lower than t and $t' \leqslant t$, we can delete that segment of the branch. In the end we end up with proofs where all branches (read upward) form bad sequences.

We can define a 'proof search tree' where we we start with the end sequent s and recursively, for existing leaves, we add as children all the premisses of rules that have the leaf as conclusion (if the new children are not already on the branch). The proof search tree is finitely-branching and all branches are finite so by Kruskal's Lemma it is finite.

Theorem. [G.-Greati-Ramanayake-St.John, 2025] Provability is decidable for $\mathbf{FL_{ekR}}$, where k is any knotted contraction rule and R is any finite set of simple rules.

Since there are no absolute bounds for the length of bad sequence in a wqo, we resort to relative length theorems: the sequences have jumps that are controlled. A *normed* wqo is a wqo $\mathbf{Q}$ endowed with a norm function $a \mapsto [a]$ into the naturals, where the preimage of every number is finite.

The branches of the proof search are bad sequences where $[a_{i+1}] \leqslant M[a_i]$, where M depends only on k and R. So, $[a_i] \leqslant M^i[s]$, for all i.

Fast-growing hierarchy and lenght functions

We consider the fast-growing hierarchy $\mathbf{F}_\alpha$ of complexity classes, where for α and ordinal less than Cantor's ε_0 (smallest ordinal such that $\varepsilon = \omega^\varepsilon$). $\mathbf{F}_\alpha$ is defined as the class of problems that have run time bounded by function in $\mathbf{F}^*_\alpha$: functions that are the composition of a single application of s function at complexity α with a function that is in a lower level than α in the Grzegorczyk hierarchy.

$\mathbf{F}_1$ is polynomial. All n-EXP are in $\mathbf{F}_2$ (elementary), which is contained in TOWER= $\mathbf{F}_3$ which is contained the union of the $\mathbf{F}_n$'s (primitive recursive), which is contained in ACK= $\mathbf{F}_\omega$, the class of Ackermann complexity.

Theorem. The class of nwqo's of the form $r\mathbb{N}^k$, with $r, k \in \mathbb{N}$ have lengths of bad sequences (as functions of the norm of their first entry) that are in $\mathbf{F}^*_\omega$.

We are also able to control the size of sequent in a branch, in terms of the height of the node and the size of the end sequent.

Theorem. Provability of $\mathbf{FL_{ekR}}$ is at most Ackermaniann, where k is any knotted contraction rule and R is any finite set of simple rules. (Also for weak commutativity.)

Theorem. The deduction theorem holds for $\mathbf{FL_{ekR}}$, where k is any knotted contraction.

Theorem. Deducibility of $\mathbf{FL_{ekR}}$ is at most Ackermaniann, where k is any knotted contraction rule and R is any finite set of simple rules. (Also for weak commutativity.)

Knotted weakening

For extensions with knotted weakening we employ a forward search.

- We start from axioms and consider larger and larger sets D_i of provable sequents, checking whether the end sequent is contained at every stage; these sets form a sequence.
- If the end sequent is not contained in any of them and there is no stabilization $D_{i+1} = D_i$, then we form a sequence $s_i \in D_{i+1} \backslash D_i$.
- We show that $s_1, s_2, \ldots$ is a bad sequence.
- We employ the length theorems to get a complexity bound.

Theorem. Provability of $\mathbf{FL_{ekR}}$ is at most Ackermaniann, where k is any knotted weakening rule and R is any finite set of simple rules. (Also for weak commutativity.)

The deduction theorem fails for knotted weakening extensions, so we cannot transfer the result to deducibility.

- Given a set S of assumption sequents (for the deduction $S \vdash s$) we design an auxiliary equivalent calculus $\mathbf{FL}_S$ in which S gets replaces by suitable inference rules.
- All of the proof-theoretic results (e.g., Curry's lemma) hold for the new calculus.
- All of the complexity results are proved to be uniform in the complexity size of S.

Theorem. Deducibility of $\mathbf{FL_{ekR}}$ is at most Ackermaniann, where k is any knotted weakening rule and R is any finite set of simple rules. (Also for weak commutativity.)

Lower bounds

We saw how to encode undecidable acceptance problems for machines to quasiequations $\&P \implies u \leqslant q_F$, where P is the set of instructions of a single machine. (This corresponds to the word problem, with input u, of the finitely-presented algebra with presentation P.)

Here, we fix the conclusion of the quasi-equations $\&P \implies q_I \leqslant q_F$ and vary the antecedent P that ranges over the instructions of machines in a given class (of Ackermannian complexity): at no time during the computation can any register value exceed the Ackermann function (on the input).

As the residuated lattices satisfy a knotted rule $x^n \leqslant x^m$, the the computations of the machine n-copies of a register can become m-copies *spontaneously*. We cannot work with square-free words, and we cannot prevent such glitches, but there is a way to detect them: We use an auxiliary *budget* register that is always equal to the (maximum) Ackermann value of the input. (Resilience.)

Theorem. Our class of machines terminates iff RL satisfies the quasiequation

$$\&P \Rightarrow q_I \leqslant q_F$$

Main Theorem. Deducibility of $\mathbf{FL}+$ a knotted rule + a (weak) commutativity is Ackermann-complete. In the case of a knotted contraction rule and commutativity, the same holds for the equational theory.

Beyond sequent rules

We mentioned that, by results of [G.-Jipsen, TAMS, 2013], $\{\vee, \cdot, 1\}$-equations give rise to analytic structural *sequent* rules (cut elimination holds).

By results of [G.-Ciabbatoni-Terui, LICS, 2008] and [G.-Ciabbatoni-Terui, APAL, 2012] strongly analytic sequent rules are essentially defined only by $\{\vee, \cdot, 1\}$-equations.

A hypersequent is a multiset $s_1 \mid \cdots \mid s_m$ of sequents s_i. Hypersequent structural rules:

$$\frac{H \mid s'_1 \quad H \mid s'_2 \quad \ldots \quad H \mid s'_n}{H \mid s_1 \mid \cdots \mid s_m}$$

Hypersequent calculi allow for the proof-theoretic study of many more extensions, such as the Gödel-Dummet logic modeled by $(x \to y) \vee (y \to x)$, as $\mid$ is a form of disjunction.

We make heavy use of results in a series of papers on *Algebraic Proof Theory* by G.-Ciabbatoni-Terui: [LICS, 2008], [AU, 2011], [APAL, 2012], [APAL, 2017].

(i) Hypersequents allow access to *finitely subdirectly irreducible* algebras in the variety and to $\mathsf{HSP_U}$-classes (positive universal classes).

(ii) Full description of analytic (hyper)sequent rules and a transformation proceedure.

(iii) The *substructural hierarchy* (similar to the arithmetical hierarchy) is defined by alternations of *positive* and *negative* connectives.

Bi-modules

Let's assume that $P = N$ is the underlying set of a residuated lattice.

- $x \cdot 1 = x = 1 \cdot x$, $(xy)z = x(yz)$
- $x(y \vee z) = xy \vee xz$ and $(y \vee z)x = yx \vee zx$

So, $(P, \vee, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

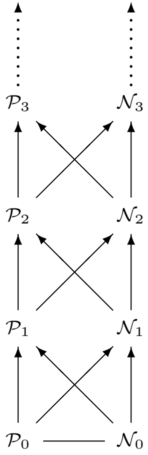
- $x\backslash(y \wedge z) = (x\backslash y) \wedge (x\backslash z)$ and $(y \wedge z)/x = (y/x) \wedge (z/x)$
- $(y \vee z)\backslash x = (y\backslash x) \wedge (z\backslash x)$ and $x/(y \vee z) = (x/y) \wedge (x/z)$
- $x\backslash(y/z) = (x\backslash y)/z$
- $1\backslash x = x = x/1$
- $(yz)\backslash x = z\backslash(y\backslash x)$ and $x/(zy) = (x/y)/z$

So, $(P, \vee, \cdot, 1)$ acts on both sides on $(N, \wedge)$ by $p \star n = n/p$ and $n \star p = p\backslash n$. Thus, $((N, \wedge), \star)$ becomes a $(P, \vee, \cdot, 1)$-bimodule. This split matches the notion of *polarity*. It also extends to $\bigvee$, $\bigwedge$.

The bimodule can be viewed as a two-sorted algebra $(P, \vee, \cdot, 1, N, \wedge, \backslash, /)$.

The absolutely free algebra for $P = N$ generated by $P_0 = N_0 = Var$ (the set of propositional variables) gives the set of all formulas. The steps of the generation process yield the *substructural hierarchy*.

Substructural hierarchy



- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined by:
  - (0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables
  - (P1) $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$
  - (P2) $a, b \in \mathcal{P}_{n+1} \Rightarrow a \vee b, a \cdot b, 1 \in \mathcal{P}_{n+1}$
  - (N1) $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$
  - (N2) $a, b \in \mathcal{N}_{n+1} \Rightarrow a \wedge b \in \mathcal{N}_{n+1}$
  - (N3) $a \in \mathcal{P}_{n+1}, b \in \mathcal{N}_{n+1} \Rightarrow a\backslash b, b/a, 0 \in \mathcal{N}_{n+1}$
- $\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\vee, \prod}$; $\mathcal{N}_{n+1} = \langle \mathcal{P}_n \rangle_{\wedge, \mathcal{P}_{n+1} \backslash, /\mathcal{P}_{n+1}}$
- $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \mathcal{N}_n \subseteq \mathcal{N}_{n+1}, \bigcup \mathcal{P}_n = \bigcup \mathcal{N}_n = Fm$
- $\mathcal{P}_1$-reduced: $\bigvee \prod p_i$
- $\mathcal{N}_1$-reduced: $\bigwedge (p_1 p_2 \cdots p_n \backslash r / q_1 q_2 \cdots q_m)$
  $p_1 p_2 \cdots p_n q_1 q_2 \cdots q_m \leqslant r$
- Sequent: $a_1, a_2, \ldots, a_n \Rightarrow a_0$ $(a_i \in Fm)$

Theorem: Deducibility of $\mathbf{FL}+$ a (weak) commutativity $+$ a knotted $+$ any finite set of $\mathcal{P}_3$ formulas is in hyper-ACK $\mathbf{F}_{\omega^\omega}$.

More, new complexity results are given in [G.-Greati-Ramanayake-St.John, 2025 (192pp)]:

	Logic(s)	Provability			Deducibility		
		Decidability	LB	UB	Decidability	LB	UB
Base logics	$\mathbf{FL_e}$	FMP[113] PS[114]	PSPACE [54]	PSPACE [54]	N[93]	–	–
	$\mathbf{FL_{ew}}$	FMP[113] PS[114]	PSPACE [54]	PSPACE [54]	FEP[52] PS(6.5)[b]	TOWER [96]	TOWER [96]
	$\mathbf{FL_{ec}}$	FMP[113] PS[115]	$\mathbf{F}_\omega$[58]	$\mathbf{F}_\omega$[58]	FEP[52] PS[115][a]	$\mathbf{F}_\omega$[58]	$\mathbf{F}_\omega$[58][a]
	$\mathbf{FL_{ec(m,1)}}, m > 2$	FEP[52] PS[53]	$\mathbf{F}_\omega$(10.4)[a]	$\mathbf{F}_\omega$(5.20)	FEP[52] PS[53]	$\mathbf{F}_\omega$(10.4)	$\mathbf{F}_\omega$(5.20)
	$\mathbf{FL_{ec(m,n)}}, n \geq 2$	FEP[52] PS(5.19)	$\mathbf{F}_\omega$(10.4)[a]	$\mathbf{F}_\omega$(5.20)	FEP[52] PS(5.19)	$\mathbf{F}_\omega$(10.4)	$\mathbf{F}_\omega$(5.20)
	$\mathbf{FL_{e(\vec{a})c(m,n)}}$	FEP[56] PS(7.30)	PSPACE [54]	$\mathbf{F}_\omega$(7.31)	FEP[56] PS(7.30)	$\mathbf{F}_\omega$(10.4)	$\mathbf{F}_\omega$(7.31)
	$\mathbf{FL_{ew(1,n)}}, n \geq 2$	FEP[52] PS[53]	PSPACE [54]	PSPACE [54]	FEP[52] PS(6.5)	$\mathbf{F}_\omega$(11.15)	$\mathbf{F}_\omega$(6.14)
	$\mathbf{FL_{ew(m,n)}}, m \geq 2$	FEP[52] PS(6.5)	PSPACE [54]	$\mathbf{F}_\omega$(6.14)	FEP[52] PS(6.5)	$\mathbf{F}_\omega$(11.15)	$\mathbf{F}_\omega$(6.14)
	$\mathbf{FL_{e(\vec{a})w(m,n)}}$	FEP[56] PS(7.33)	PSPACE [54]	$\mathbf{F}_\omega$(7.35)	FEP[56] PS(7.33)	$\mathbf{F}_\omega$(11.15)	$\mathbf{F}_\omega$(7.35)
	$\mathbf{FL_i}$	FMP[113] PS[114]	PSPACE [54]	PSPACE [54]	FEP[105] PS[59]	$\mathbf{F}_{\omega^\omega}$[59]	$\mathbf{F}_{\omega^\omega}$[59]
	$\mathbf{FL_{c(m,n)}}$	N[60]	–	–	N[94]	–	–
	$\mathbf{FL_{w(1,2)}}$	FMP[67] PS[54]	PSPACE [54]	PSPACE [54]	FEP[107]	open	open
	$\mathbf{FL_{w(1,n)}}$	FMP[67] PS[54]	PSPACE [54]	PSPACE [54]	open	open	open
	$\mathbf{FL_{w(m,n)}}, m > 1$	open	PSPACE [54]	open	N[94]	–	–
$\mathcal{A} \subseteq \mathcal{N}_2$	$\mathbf{FL_{ec}}(\mathcal{A})$	FEP[56] PS[66]	$\mathbf{F}_\omega$(10.12)[c]	$\mathbf{F}_\omega$[65]	FEP[56] PS[65][a]	$\mathbf{F}_\omega$(10.4)[c]	$\mathbf{F}_\omega$[65][a]
	$\mathbf{FL_{ec(m,n)}}(\mathcal{A})$	FEP[56] PS(5.19)	$\mathbf{F}_\omega$(10.12)[c]	$\mathbf{F}_\omega$(5.20)	FEP[56] PS(5.19)	$\mathbf{F}_\omega$(10.4)[c]	$\mathbf{F}_\omega$(5.20)
	$\mathbf{FL_{e(\vec{a})c(m,n)}}(\mathcal{A})$	FEP[56] PS(7.30)	$\mathbf{F}_\omega$(10.12)[c]	$\mathbf{F}_\omega$(7.31)	FEP[56] PS(7.30)	$\mathbf{F}_\omega$(10.4)[c]	$\mathbf{F}_\omega$(7.31)
	$\mathbf{FL_{ew}}(\mathcal{A})$	FEP[56] PS[65]	PSPACE [54]	$\mathbf{F}_\omega$[65]	FEP[56] PS(6.5)	$\mathbf{F}_\omega$(11.15)[c]	$\mathbf{F}_\omega$[65][a]
	$\mathbf{FL_{ew(m,n)}}(\mathcal{A})$	FEP[56] PS(6.5)	PSPACE [54]	$\mathbf{F}_\omega$(6.14)	FEP[56] PS(6.5)	$\mathbf{F}_\omega$(11.15)[c]	$\mathbf{F}_\omega$(6.14)
	$\mathbf{FL_{e(\vec{a})w(m,n)}}(\mathcal{A})$	FEP[56] PS(7.33)	PSPACE [54]	$\mathbf{F}_\omega$(7.35)	FEP[56] PS(7.33)	$\mathbf{F}_\omega$(11.15)[c]	$\mathbf{F}_\omega$(7.35)
	$\mathbf{FL_i}(\mathcal{A})$	FEP[67] PS[59]	PSPACE [54]	$\mathbf{F}_{\omega^\omega}$[59]	FEP[67] PS[59]	PSPACE [54]	$\mathbf{F}_{\omega^\omega}$[59]
$\mathcal{A} \subseteq \mathcal{P}_3^\flat$	$\mathbf{FL_{ec}}(\mathcal{A})$	FEP(3.3) PS[65]	open	$\mathbf{F}_{\omega^\omega}$[65]	FEP(3.3) PS[65][a]	open	$\mathbf{F}_{\omega^\omega}$[65][a]
	$\mathbf{FL_{ec(m,n)}}(\mathcal{A})$	FEP(3.3) PS(5.19)	open	$\mathbf{F}_{\omega^\omega}$(5.20)	FEP(3.3) PS(5.19)	open	$\mathbf{F}_{\omega^\omega}$(5.20)
	$\mathbf{FL_{e(\vec{a})c(m,n)}}(\mathcal{A})$	FEP(3.3) PS(7.30)	open	$\mathbf{F}_{\omega^\omega}$(7.31)	FEP(3.3) PS(7.30)	open	$\mathbf{F}_{\omega^\omega}$(7.31)
	$\mathbf{FL_{ew}}(\mathcal{A})$	FEP(3.3) PS[65]	open	$\mathbf{F}_{\omega^\omega}$[65]	FEP(3.3) PS(6.5)	open	$\mathbf{F}_{\omega^\omega}$[65][a]
	$\mathbf{FL_{ew(m,n)}}(\mathcal{A})$	FEP(3.3) PS(6.5)	open	$\mathbf{F}_{\omega^\omega}$(6.14)	FEP(3.3) PS(6.5)	open	$\mathbf{F}_{\omega^\omega}$(6.14)
	$\mathbf{FL_{e(\vec{a})w(m,n)}}(\mathcal{A})$	FEP(3.3) PS(7.33)	open	$\mathbf{F}_{\omega^\omega}$(7.35)	FEP(3.3) PS(7.33)	open	$\mathbf{F}_{\omega^\omega}$(7.35)
	$\mathbf{FL_i}(\mathcal{A})$	FEP(3.3) PS(8.13)	open	$\mathbf{F}_{\omega^{\omega^{\omega^\omega}}}$(8.20)	FEP(3.3) PS(8.13)	open	$\mathbf{F}_{\omega^{\omega^{\omega^\omega}}}$(8.20)

Capturing more varieties

Distributive residuated lattices (and various subvarieties):

- proof theory and decidability [Giambrone JPL 185], [Brady JPL 1990]
- residated frames [G.-Jipsen AU 2017]
- decidability and FMP [Kozak SL 2009]
- FEP [G.-Cardona AU 2017]

Involutive residuated lattices:

- proof theory and FMP (also for n-periodic ones) [G.-Jipsen TAMS 2013]
- frames and completions [G.-Prenosil JoA 2023]

ℓ-groups: proof theory and complexity (coNP-complete) [G.-Metcalfe, APAL 2016]

Using the method of *diagrams*:

- ℓ-groups: decidability [Holland and McCleary HJM 1979]
- Distributive ℓ-monoids: decidability [G.-Colacito, Metcalfe, Santchi JoA 2022]
- Distributive ℓ-pregroups: decidability and complexity [G.-Gallardo JoA 2024]
- Periodic ℓ-pregroups: decidability and complexity [G.-Gallardo JoA 2025]
- Weakening relation algebras over ordinals: decidability

Thank you!!