Varieties of MV-monoids and positive MV-algebras

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Introduction

An MV-monoid is an algebra $\langle A, \vee, \wedge, \oplus, \odot, 0, 1\rangle$ where

1. $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice;
2. $\langle A, \oplus, 0\rangle$ and $\langle A, \odot, 1\rangle$ are commutative monoids;
3. $\oplus$ and $\odot$ distribute over $\vee$ and $\wedge$;
4. for every $x, y, z \in A$,

$$(x \oplus y) \odot ((x \odot y) \oplus z) = (x \odot (y \oplus z)) \oplus (y \odot z);$$

$$(x \odot y) \oplus ((x \oplus y) \odot z) = (x \oplus (y \odot z)) \odot (y \oplus z);$$

$$(x \odot y) \oplus z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \vee z;$$

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The class MV^+ of positive MV-algebras is the proper subquasivariety of MVM, axiomatized relatively to MVM by

$$(x \oplus z \approx y \oplus z \text{ and } x \odot z \approx y \odot z) \implies x \approx y.$$

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A commutative ℓ-monoid is an algebra $\mathbf{M} = \langle M, \vee, \wedge, +, 0 \rangle$ with the following properties:

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A unital commutative ℓ-monoid is an algebra $\langle M, \vee, \wedge, +, 1, 0, -1\rangle$ with the following properties:

1. $\langle M, \vee, \wedge, +, 0\rangle$ is a commutative ℓ-monoid;
2. $-1 + 1 = 0$;
3. $-1 \leq 0 \leq 1$;
4. for all $x \in M$ there is $n \in \mathbb{N}$ such that

$$\underbrace{(-1) + \cdots + (-1)}_{n \text{ times}} \leq x \leq \underbrace{1 + \cdots + 1}_{n \text{ times}}.$$

Introduction

For every unital commutative ℓ-monoid $\mathbf{M}$, its unit interval can be equipped with the structure of an MV-monoid:

$$\Gamma(\mathbf{M}) = \langle \Gamma(\mathbf{M}), \vee, \wedge, \oplus, \odot, 0, 1 \rangle$$

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$$\Gamma(\mathbf{M}) \coloneqq \{x \in M \mid 0 \leq x \leq 1\}$$

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The assignment $\mathbf{M} \mapsto \Gamma(\mathbf{M})$ can be extended to morphisms to define a functor

$$\Gamma \colon \mathsf{u\ell M} \to \mathsf{MVM}$$

from the category uℓM of unital commutative ℓ-monoids to the category MVM of MV-monoids (Cat. equivalence Abbadini '21).

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Theorem (Abbadini, Jipsen, Kroupa, Vannucci '22)

$\Gamma \colon \mathsf{u\ell M} \to \mathsf{MVM}$ restricts to an equivalence between cancellative unital commutative ℓ-monoids and positive MV-algebras (MV^+).

Introduction

Examples:

- Ł_n^+, the reduct of the MV-algebra with universe $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ (i.e. $\Gamma(\frac{1}{n}\mathbb{Z}) = \text{Ł}_n^+$);

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- $\mathbf{C}_2^\Delta$, the unique MV-monoid on the 3-element chain $0 < \varepsilon < 1$ satisfying $\varepsilon \oplus \varepsilon = \varepsilon$ and $\varepsilon \odot \varepsilon = 0$;
- $\mathbf{C}_2^\nabla$, the dual of $\mathbf{C}_2^\Delta$;
- $\mathbf{C}_3^\Delta$, the unique MV-monoid on $0 < \varepsilon < 2\varepsilon < 1$ with $\varepsilon \oplus \varepsilon = 2\varepsilon$ and $2\varepsilon \odot 2\varepsilon = 0$ (not a positive MV-algebra).

Subdirectly irreducible MV-monoids

Theorem (Abbadini, Aglianò, SF) Let **M** be a unital commutative ℓ-monoid. TFAE:

1. **M** is totally ordered;
2. $\Gamma(\mathbf{M})$ is totally ordered, and, for every $x, y \in \Gamma(\mathbf{M})$, $x \oplus y = 1$ or $x \odot y = 0$.

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Theorem (Abbadini, Aglianò, SF) If an MV-monoid **A** is subdirectly irreducible, then it is nontrivial, totally ordered, and such that, for all $x, y \in A$, $x \oplus y = 1$ or $x \odot y = 0$.

The almost minimal varieties of MV-monoids

Theorem (Abbadini, Aglianò, SF) The almost minimal subvarieties of MVM are precisely $\mathcal{V}(\mathbf{C}_2^{\Delta})$, $\mathcal{V}(\mathbf{C}_2^{\nabla})$ and $\mathcal{V}(\mathbf{Ł}_p^{+})$ (for p prime), and they are all pairwise distinct.

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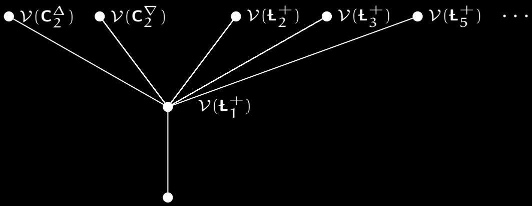

Figure: The bottom part of $\Lambda(\mathrm{MVM})$

MV-monoids of small cardinalities

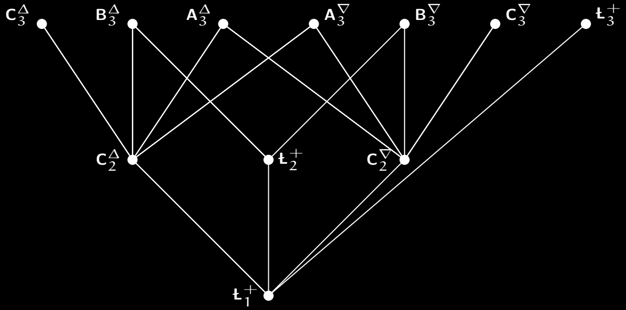

Figure: Subdirectly irreducible MV-monoids with cardinality ≤ 4 ordered by: $\mathbf{A} \leq \mathbf{B}$ iff $\mathbf{A} \in \mathbb{HS}(\mathbf{B})$

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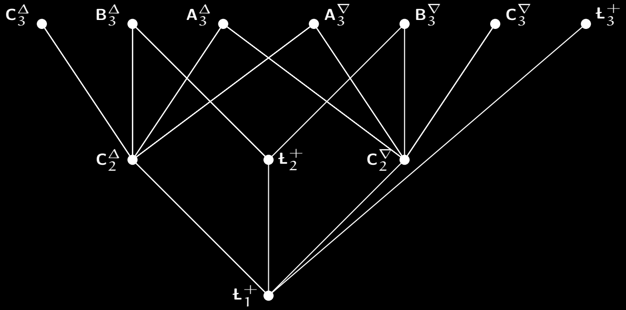

Figure: Subdirectly irreducible MV-monoids with cardinality ≤ 4 ordered by: $\mathbf{A} \leq \mathbf{B}$ iff $\mathbf{A} \in \mathbb{HS}(\mathbf{B})$

Theorem (Abbadini, Aglianò, SF) The ideal of $\Lambda(\mathsf{MVM})$ consisting of all varieties generated by MV-monoids with at most 4 elements is isomorphic to the lattice of downward-closed subsets of the previous poset.

Positive MV-algebras

Theorem (Abbadini, Aglianò, SF) For a finite positive MV-algebra $\mathbf{A}$, TFAE:

1. $\mathbf{A}$ is subdirectly irreducible;
2. $\mathbf{A} \cong \mathbf{Ł}_n^+$ for some $n \in \mathbb{N} \setminus \{0\}$;
3. $\mathbf{A}$ is nontrivial, totally ordered and, for all $x, y \in A$, either $x \oplus y = 1$ or $x \odot y = 0$.

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Theorem (Abbadini, Aglianò, SF) The varieties of positive MV-algebras are precisely the varieties generated by a finite subset of $\{\mathbf{Ł}_n^+ \mid n \in \mathbb{N} \setminus \{0\}\}$.

Positive MV-algebras

We call divisor-closed finite set a finite subset I of $\mathbb{N} \setminus \{0\}$ such that, for every $n \in I$ and $k \in \mathbb{N} \setminus \{0\}$, if k divides n, then $k \in I$.

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For a divisor-closed finite set I, we set

$$\mathcal{K}_I := \{\text{Ł}_n^+ \in \mathsf{MV}^+ \mid n \in I\}.$$

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Theorem (Abbadini, Aglianò, SF) The set $\Lambda(\mathsf{MV}^+)$ of varieties of positive MV-algebras is in bijection with the set $\mathcal{J}$ of divisor-closed finite sets, as witnessed by:

$$\begin{aligned} f\colon \mathcal{J} &\longrightarrow \Lambda(\mathsf{MV}^+) \\ I &\longmapsto \mathcal{V}(\mathcal{K}_I) \end{aligned} \qquad \begin{aligned} g\colon \Lambda(\mathsf{MV}^+) &\longrightarrow \mathcal{J} \\ \mathcal{V} &\longmapsto \{n \in \mathbb{N} \setminus \{0\} \mid \text{Ł}_n^+ \in \mathcal{V}\}. \end{aligned}$$

Axiomatizations

$$\tau_{0,k}(x) \coloneqq \begin{cases} 1 & \text{if } k \leq -1, \\ 0 & \text{if } k \geq 0. \end{cases}$$

The inductive case is as follows:

$$\tau_{n+1,k}(x) = \tau_{n,k-1}(x) \odot (x \oplus \tau_{n,k}(x)),$$

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and

$$\tau_{2,k}(x) = \begin{cases} 1 & \text{if } k \leq -1, \\ x \oplus x & \text{if } k = 0, \\ x \odot x & \text{if } k = 1, \\ 0 & \text{if } k \geq 2, \end{cases}$$

Axiomatizations

For $n \in \mathbb{N}$, let Φ_n be the following set of equations:

$$\begin{aligned} \tau_{n,k}(x) \oplus \tau_{n,k}(x) &\approx \tau_{n,k}(x) \\ \tau_{n,k}(x) \odot \tau_{n,k}(x) &\approx \tau_{n,k}(x). \end{aligned} \tag{1}$$

with $k \in \{0, \ldots, n-1\}$ (i.e. $\tau_{n,k}(x)$ is idempotent).

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Given a divisor-closed set I with maximum m (set $m = 0$ if $I = \emptyset$). We define Σ_I as:

$$(m+1)x \approx mx \tag{2}$$

union

$$m((k-1)x)^k \approx (kx)^m \tag{3}$$

for all $1 \leq k \leq m$ with $k \notin I$.

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Theorem (Abbadini, Aglianò, SF) Let I be a divisor-closed finite set; then $\mathcal{V}(\mathcal{K}_I)$ is axiomatized by $\Phi_{\mathrm{lcm}(I)} \cup \Sigma_I$ relatively to the variety of MV-monoids.

Axiomatizations

Example: Let $I = \{1, 2, 3\}$ and $J = \{1, 2, 3, 6\}$.

$\mathcal{V}(\mathcal{K}_I)$ axioms: $\mathsf{MVM} \cup \Phi_6 \cup \{4x \approx 3x\}$;

$\mathcal{V}(\mathcal{K}_J)$ axioms:

$\mathsf{MVM} \cup \Phi_6 \cup \{7x \approx 6x, 6(3x)^4 \approx (4x)^6, 6(4x)^5 \approx (5x)^6\}$.

Note that the failure of $4x \approx 3x$ in $\mathbf{Ł}_6^+$ is witnessed by $\frac{1}{6}$.

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Variety	**Axiomatization**
$\mathcal{V}(\mathbf{C}_2^{\Delta})$	$x \oplus x \approx x$
$\mathcal{V}(\mathbf{C}_2^{\nabla})$	$x \odot x \approx x$
$\mathcal{V}(\mathbf{L}_1^{+})$	$x \oplus x \approx x$ and $x \odot x \approx x$
$\mathcal{V}(\text{Ł}_n^{+})$	$\tau_{n,k}(x) \oplus \tau_{n,k}(x) \approx \tau_{n,k}(x)$ (for $0 \leq k \leq n-1$) $\tau_{n,k}(x) \odot \tau_{n,k}(x) \approx \tau_{n,k}(x)$ (for $0 \leq k \leq n-1$)
$\mathcal{V}(\{\text{Ł}_n^{+} \mid n \in I\})$ (I div.-closed fin. set)	(setting $l :: \operatorname{lcm} I$ and $m :: \max I$) $\tau_{l,k}(x) \oplus \tau_{l,k}(x) \approx \tau_{l,k}(x)$ (for $0 \leq k \leq l-1$) $\tau_{l,k}(x) \odot \tau_{l,k}(x) \approx \tau_{l,k}(x)$ (for $0 \leq k \leq l-1$) $(m+1)x \approx mx$ $m((k-1)x)^k \approx (kx)^m$ (for $1 \leq k \leq m$ s.t. $k \notin I$)

Hölder's theorem for unital commutative ℓ-monoids

Theorem (Abbadini, Aglianò, SF) Let $\mathbf{M}$ be a nontrivial totally ordered unital commutative ℓ-monoid. There is a unique homomorphism from $\mathbf{M}$ to $\mathbb{R}$.

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A unital commutative ℓ-monoid **M** is Archimedean provided that, for all $x, y \in M$, if for all $n \in \mathbb{N}$ we have $nx \leq ny + 1$, then $x \leq y$.

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Theorem (Abbadini, Aglianò, SF) [Hölder's theorem for unital commutative ℓ-monoids] Let **M** be an Archimedean nontrivial totally ordered unital commutative ℓ-monoid. The unique homomorphism from **M** to $\mathbb{R}$ is injective, thus **M** is isomorphic to a subalgebra of $\mathbb{R}$.

Bibliography

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Thank you for your attention