Pure Embeddings in Acts: Stability and Cofibrant Generation

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Motivation

- Abstract elementary classes (AECs) generalize first-order model theory.
- Research in AECs of modules has been very fruitful.
- Stability and cofibrant generation have been shown for modules with pure monomorphisms.
- Acts are a natural generalization of modules.

Preliminaries: Acts

S-act

For a monoid S, an *S-act* is a set A together with a multiplication $S \times A \to A$ such that $1a = a$ and $(st)a = s(ta)$ $\forall a \in A,\ s, t \in S$, i.e., "a module without additive structure."

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LO monoid

S is an *LO* monoid if, for every $s, t \in S$, either $s \in St$ or $t \in Ss$.

Examples: $(\mathbb{N}, +)$ is LO, but $(\mathbb{Z}^+, \cdot)$ is not.

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Theorem (Mustafin 1988)

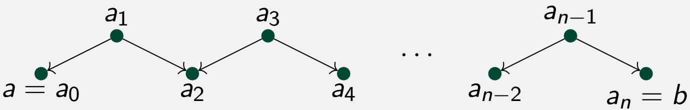
$Th(A)$ is stable for every S-act A if and only if S is LO.

This contrasts with module theory, where it is know that every first-order theory of modules is stable.

Preliminaries: Acts

Connected Outside

For S-acts $M \subseteq A$, we say that $a, b \in A \setminus M$ are *connected outside M* if there are $a_0, \ldots, a_n \in A \setminus M$ such that $a = a_0$, $b = a_n$, and either $a_i \in Sa_{i+1}$ or $a_{i+1} \in Sa_i$ $\forall\ 0 \leq i \leq n-1$.



all outside of M

Remark

$C^A_M(a) = \{b \in A \setminus M \mid a \text{ and } b \text{ are connected outside } M\}$ is an analogue of group orbits and partitions $A \setminus M$ into "connected components".

Preliminaries: Purity

Pure Subact

For S-acts $A \subseteq B$, we say that A is a *pure* subact of B ($A \leq_p B$) if every finite system of equations of the forms

(I) $sx = tx$ for $s, t \in S$

(II) $sx = ty$ for $s, t \in S$

(III) $sx = a$ for $a \in A$

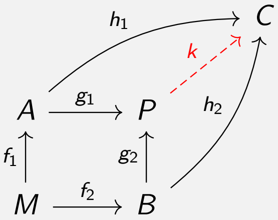
is solvable in B if and only if it is solvable in A.

- Example: If $S = (\mathbb{N}, +)$, then $\mathbb{N} \leq_p \mathbb{N} + \frac{1}{2}\mathbb{N}$ via $n \mapsto \lfloor n \rfloor$.
- Non-Example: If $S = (\mathbb{N}, +)$, then $\mathbb{N} \not\leq_p \mathbb{Z}$ because of $1 + x = 0$.

Preliminaries: Pushouts

Pushout

In a category $\mathcal{K}$, the *pushout* of a pair of arrows (f_1, f_2) is an object P together with arrows (g_1, g_2) such that $g_1 f_1 = g_2 f_2$ and, whenever (h_1, h_2) satisfy $h_1 f_1 = h_2 f_2$ there is a unique arrow $k : P \to C$ making the diagram commute.



In $(S\text{-Act}, \text{pure})$, we have $P = (A \coprod B)/\sim$ where $\sim$ identifies the copies of M in A and B, respectively.

Preliminaries: Pushouts Cont.

Pushouts in (S-Act, pure)

For S-acts $M \leq_p A, B$,

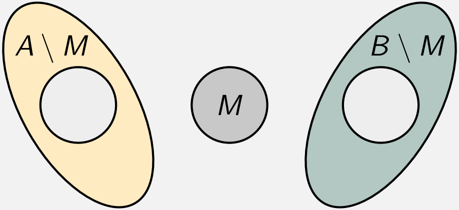
$$P = (A \setminus M) \times \{1\} \cup M \times \{0\} \cup (B \setminus M) \times \{2\}$$

with

$$g_1(A) = (A \setminus M) \times \{1\} \cup M \times \{0\}$$

and

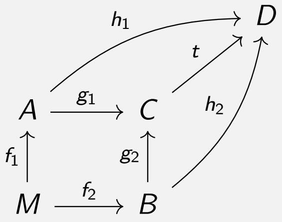
$$g_2(B) = (B \setminus M) \times \{2\} \cup M \times \{0\}.$$



Preliminaries: Independence Relations

Independence Relation (Lieberman-Rosický-Vasey 2019)

An *independence relation* on a category $\mathcal{K}$ is a set $\smile\!\!\!\!\!|\,$ of commutative squares of arrows in $\mathcal{K}$ such that, for any commutative diagram



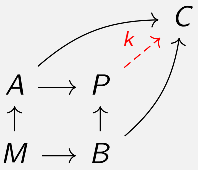
we have that $(f_1, f_2, g_1, g_2) \in \smile\!\!\!\!\!|\,$ if and only if $(f_1, f_2, h_1, h_2) \in \smile\!\!\!\!\!|\,$.
In this case, we call (f_1, f_2, g_1, g_2) an *independent square*.

We write $A \underset{M}{\overset{C}{\smile\!\!\!\!\!|\,}} B$ if and only if $(i_{MA}, i_{MB}, i_{AC}, i_{BC}) \in \smile\!\!\!\!\!|\,$.

Some Independence Relations in (S-Act, pure)

Pure-Effective Squares $\mathop{\smile\!\!\!\!\!\!\mid}_p$

$A \underset{M}{\overset{C}{\mathop{\smile\!\!\!\!\!\!\mid}}}_p B$ if and only if k is a pure embedding:



Disconnected Pullback Squares $\mathop{\smile\!\!\!\!\!\!\mid}_{dc}$

$A \underset{M}{\overset{C}{\mathop{\smile\!\!\!\!\!\!\mid}}}_{dc} B$ if and only if

- $A \cap B = M$, and
- $\mathrm{C}^C_M(A \setminus M) \cap \mathrm{C}^C_M(B \setminus M) = \emptyset$.

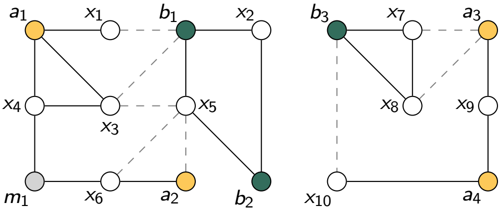
Key Lemma

Lemma

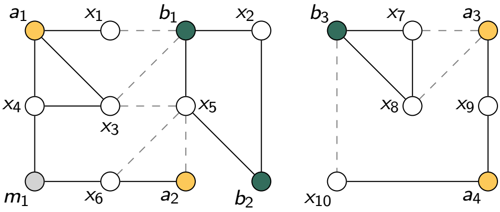
In (S-Act, pure), we have $\perp_{dc} \subseteq \perp_p$.

Proof.

Let Σ be a system of equations in variables $\{x_1, \ldots, x_n\}$ with constants from P and solution $\{c_1, \ldots, c_n\}$ in C. Associate to Σ a graph G with vertices $P \cup \{x_1, \ldots, x_n\}$.



Key Lemma Cont.



Since $C^C_M(A \setminus M) \cap C^C_M(B \setminus M) = \emptyset$, no $a \in A \setminus M$ and $b \in B \setminus M$ are in the same connected component. For equations $(s_\ell x_{i_\ell} = t_\ell x_{j_\ell}) \in \Sigma$ with x_{i_ℓ} connected to $A \setminus M$ in G and x_{j_ℓ} not, we know x_{i_ℓ} and x_{j_ℓ} are not in the same connected component of G. Thus this is one of the edges we removed, and so $s_\ell c_{i_\ell} = t_\ell c_{j_\ell} = m_\ell \in M$.

Key Lemma Cont.

Define the new systems of equations

$$\Delta_A = \{\varphi \in \Sigma \mid \mathrm{var}(\varphi) \text{ are connected to } A \setminus M \text{ in } G\} \cup \{s_\ell x_{i_\ell} = m_\ell \mid 1 \leq \ell \leq k\}$$

and

$$\Delta_B = \{\varphi \in \Sigma \mid \mathrm{var}(\varphi) \text{ is not connected to } A \setminus M \text{ in } G\} \cup \{t_\ell x_{j_\ell} = m_\ell \mid 1 \leq \ell \leq k\}.$$

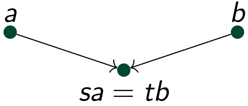
Since $A \to C$ and $B \to C$ are pure, Δ_A and Δ_B have solutions $\bar{a} \in A$ and $\bar{b} \in B$, respectively. Then $\bar{a} \cup \bar{b} \in P$ is a solution to Σ since, either $\varphi \in \Delta_A \cup \Delta_B$ or $\varphi = (s_\ell x_{i_\ell} = t_\ell x_{j_\ell})$ and so $s_\ell a_{i_\ell} = m_\ell = t_\ell b_{j_\ell}$. □

Equality of Independence Relations

Corollary

If S is LO, then $\mathop{\smile}\limits^{|}_{dc} = \mathop{\smile}\limits^{|}_{p}$.

Proof. When S is LO, every connecting path from a to b reduces to $sa = tb$ for some $s, t \in S$.



Thus $A \cap B = M$ implies that $\mathrm{C}_M^C(A \setminus M) \cap \mathrm{C}_M^C(B \setminus M) = \emptyset$ while $A \cap B = M$ follows from the injectivity of k. □

Properties of Independence Relations

Stable Independence Relation (Lieberman-Rosický-Vasey 2019)

We say that $\perp$ is a *stable* independence relation if it has:

- Existence: Every span $A \leftarrow M \rightarrow B$ can be completed into an independent square.
- Uniqueness: Every span has a unique (up to equivalence) independent square.
- Symmetry: Independent squares can be reflected across the positive diagonal.
- Transitivity: Independent squares are closed under composition.
- Local Character: Given $A \rightarrow C \leftarrow B$, there is a small S-act that creates an independent square.
- Witness Property: A commutative square can be verified to be independent using only small sets.

Main Result

Theorem

The following are equivalent:

1. *S is LO.*
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    - *Key model theoretic dividing line bounding number of types*

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Remark

- $(R\text{-Mod}, \leq_p)$ is always stable.

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3. $(S\text{-}Act, \leq_p)$ *is stable.*
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4. *Pure embeddings are cofibrantly generated in S-Act.*
   - *All arrows can be generated from a set.*

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The following are equivalent:

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3. $(S\text{-}Act, \leq_p)$ *is stable.*
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4. *Pure embeddings are cofibrantly generated in S-Act.*
    - *All arrows can be generated from a set.*

Remark

- $(R\text{-Mod}, \leq_p)$ is always stable.
- Pure embeddings are always cofibrantly generated in R-Mod.

Summary and Future Work

Summary

- S-Act and R-Mod have fundamental differences with regards to pure embeddings.
- Cofibrant generation of pure embeddings implies there are enough pure injectives.
- Everything we've done generalizes to presheaf categories.

Future Work

- (S-Act, $\leq_p$) as an AEC
  - Stability Cardinals
  - Superstability
  - Galois Types
- Other classes of Acts
- Applications in Acts Theory

References

THANK YOU!

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