Commuting degree of BCK-algebras

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Overview

Theorem (Gustafson, 1973)

Let G be a finite a finite group. Choose $x, y \in G$ uniformly randomly with replacement. The probability that $xy = yx$ is either

$$1 \text{ (if } G \text{ is Abelian)} \quad \text{or} \quad \leq \frac{5}{8}.$$

There is a *gap* in the possible probabilities.

One common notation for this probability is $\Pr(G)$, but I will use the notation $\mathrm{cd}(G)$, which stands for *commuting degree*.

This commuting probability is related to structural properties:

- If $\mathrm{cd}(G) = \frac{5}{8}$, then G is nilpotent.
- If G is non-Abelian simple, then $\mathrm{cd}(G) \leq \frac{1}{12}$.

There is a substantial literature about commuting probabilities in finite groups. There is a good survey by Das, Nath, and Pournaki (2013).

Commuting probabilities for other algebraic structures have also been studied:

- MacHale, 1976
  If R is a finite non-commutative ring, then $\mathrm{cd}(R) \leq \frac{5}{8}$.
- MacHale, 1990
  Among finite semigroups, the commuting probability can be as close to 1 or 0 as you want.
- Givens, 2008
  The set of all commuting probabilities for finite semigroups is dense in $(0, 1]$.
- Ponomarenko & Selinski, 2012
  The set of all commuting probabilities for finite semigroups is $(0, 1] \cap \mathbb{Q}$.

What about equations other than $xy = yx$?

Probabilities for some commutator-like equations in groups were considered by

Lescot (1995)

Delizia, Jezernik, Moravec, et al (2020)

Kocsis (2020)

The primary inspiration for the present work is a recent paper of Bumpus and Kocsis (2024) which considers probability questions for equations over Heyting algebras.

Most general version of the question:

Given an n-tuple $\mathbf{a}$ from a finite algebraic structure $\mathbf{A}$, what is the probability that $\mathbf{a}$ satisfies some given first-order formula φ?

Definition

Given a first-order language $\mathcal{L}$, a finite $\mathcal{L}$-structure $\mathbf{A}$, and an $\mathcal{L}$-formula $\varphi(x_1, x_2, \dots x_n)$ in n variables, the quantity

$$\mathrm{ds}(\varphi, \mathbf{A}) = \frac{|\{\, \mathbf{a} \in A^n \mid \varphi(\mathbf{a}) \,\}|}{|A|^n}$$

is the *degree of satisfiability* of the formula φ for the structure $\mathbf{A}$.

We say that φ has *finite satisfiability gap* ε if there is a constant $\varepsilon > 0$ such that, for every finite $\mathcal{L}$-structure $\mathbf{A}$, either

$$\mathrm{ds}(\varphi, \mathbf{A}) = 1 \text{ or}$$
$$\mathrm{ds}(\varphi, \mathbf{A}) \leq 1 - \varepsilon\,.$$

Example

Gustafson's result can be rephrased as saying the equation $xy = yx$ has finite satisfiability gap $\frac{3}{8}$ in the language of groups.

Theorem (Bumpus and Kocsis, 2024)

In the language of Heyting algebras,

- the equations $x = \top$ and $\neg x = \top$ have satisfiability gap $\frac{1}{2}$,
- the equation $x \vee \neg x = \top$ has satisfiability gap $\frac{1}{3}$.

These are the only formulas in one variable with finite satisfiability gap.

What can we say in the world of BCK-algebras?

Definition

A *BCK-algebra* is an algebra $\mathbf{A} = \langle A; \cdot, 0 \rangle$ of type $(2, 0)$ such that

1. $\big[(x \cdot y) \cdot (x \cdot z)\big] \cdot (z \cdot y) = 0$
2. $\big[x \cdot (x \cdot y)\big] \cdot y = 0$
3. $x \cdot x = 0$
4. $0 \cdot x = 0$
5. $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

for all $x, y, z \in A$.

These algebras are partially ordered by: $x \leq y$ iff $x \cdot y = 0$.

The element $x \wedge y := y \cdot (y \cdot x)$ is a lower bound for x and y.

If $x \wedge y = y \wedge x$ for all $x, y \in \mathbf{A}$, we say $\mathbf{A}$ is *commutative*.

A BCK-algebra $\mathbf{A}$ is *bounded* if there exists an element $1 \in A$ such that $x \cdot 1 = 0$ for all $x \in A$.

If **A** is bounded, we define the term operation $\neg x := 1 \cdot x$.

If **A** is bounded and commutative, we define the term operation

$$x \vee y := \neg(\neg x \wedge \neg y)\,.$$

Theorem (E., 2023)

In the language of bounded commutative BCK-algebras,

- the equations $x = 1$ and $\neg x = 1$ have satisfiability gap $\frac{1}{2}$,
- the equation $x \vee \neg x = 1$ has satisfiability gap $\frac{1}{3}$.

In that same paper, I showed several other equations in the language of BCK-algebras do *not* have a finite satisfiability gap.

Today I'll focus on the commutativity equation $x \wedge y = y \wedge x$.

Let $\mathbf{A}$ be a BCK-algebra of order n and put

$$C(\mathbf{A}) = \{\, (x, y) \in A^2 \mid x \wedge y = y \wedge x \,\}.$$

Then we define the *commuting degree* of $\mathbf{A}$ to be

$$\operatorname{cd}(\mathbf{A}) = \frac{|C(\mathbf{A})|}{|\mathbf{A}|^2} = \frac{|C(\mathbf{A})|}{n^2}.$$

Proposition (E., 2023)

Among non-commutative BCK-algebras $\mathbf{A}$ of order n, the following bounds are sharp:

$$\frac{3n-2}{n^2} \leq \operatorname{cd}(\mathbf{A}) \leq \frac{n^2-2}{n^2}.$$

Let

$$\mathcal{CD}(n) = \left\{ \frac{n^2-2}{n^2}, \frac{n^2-4}{n^2}, \ldots, \frac{3n}{n^2}, \frac{3n-2}{n^2} \right\}.$$

This is the set of possible commuting degrees.

One can check that $|\mathcal{CD}(n)| = T_{n-2}$, the $(n-2)^{\text{nd}}$ triangular number, and therefore we can rewrite $\mathcal{CD}(n)$ as:

$$\mathcal{CD}(n) = \left\{ \frac{n^2-2}{n^2}, \frac{n^2-4}{n^2}, \ldots, \frac{n^2-2T_{n-2}}{n^2} \right\} = \left\{ \frac{n^2-2k}{n^2} \right\}_{k=1}^{T_{n-2}}.$$

Empirically, I observed that every value of $\mathcal{CD}(3)$, $\mathcal{CD}(4)$, and $\mathcal{CD}(5)$ was obtained by some algebra of the corresponding order.

Theorem (E., 2025)

For every $n \geq 3$, every commuting degree in $\mathcal{CD}(n)$ is achieved by an algebra of order n.

$\cdot$	0	1
0	0	0
1	1	0

$\cdot$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

$\cdot$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Table: The algebras $\mathbf{2}$, $\mathbf{3}^{\mathsf{p}}$, and $\mathbf{3}^{\mathsf{c}}$

$\mathbf{2}$ is the unique BCK-algebra of order 2. It is comm. and P.I.

$\mathbf{3}^{\mathsf{p}}$ is positive implicative but not commutative

$\mathbf{3}^{\mathsf{c}}$ is commutative but not positive implicative

Two constructions

1. Given any BCK-algebra $\mathbf{A}$ of order $n-1$, we construct a new BCK-algebra of order n by appending a new top element, call it $\top$, and extending the BCK-operation as follows:

$$x \cdot \top = 0$$
$$\top \cdot \top = 0$$
$$\top \cdot x = \top$$

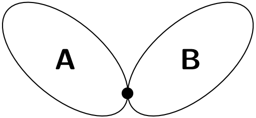
for all $x \in \mathbf{A}$.
This is known as *Iséki's extension of* $\mathbf{A}$, which we will denote

$$\mathbf{A} \oplus \top .$$

We note that $\mathbf{3}^{\mathrm{p}} \cong \mathbf{2} \oplus \top$.

Two constructions

2. Given two BCK-algebras **A** and **B**, we build their *BCK-union*, denoted $\mathbf{A} \sqcup \mathbf{B}$, by identifying $0_{\mathbf{A}}$ and $0_{\mathbf{B}}$ but keeping them otherwise disjoint.



The operation is defined by $x \cdot y = \begin{cases} x \cdot_{\mathbf{A}} y & \text{if } x, y \in A \\ x \cdot_{\mathbf{B}} y & \text{if } x, y, \in B \\ x & \text{otherwise} \end{cases}$.

Note that if **A** has order $n-1$, then $\mathbf{A} \sqcup \mathbf{2}$ has order n.

Obtaining values in $\mathcal{CD}(n)$

Proposition (E., 2023)

If $\mathbf{A}$ is a BCK-algebra of order n with $\text{cd}(\mathbf{A}) = \frac{k}{n^2}$, then

$$\text{cd}(\mathbf{A} \oplus \top) = \frac{k+3}{(n+1)^2} \qquad \text{cd}(\mathbf{A} \sqcup \mathbf{2}) = \frac{k+2n+1}{(n+1)^2}$$

From the algebras $\mathbf{2}$, $\mathbf{3}^{\mathsf{p}}$, and $\mathbf{3}^{\mathsf{c}}$, we can obtain all values of $\mathcal{CD}(n)$ by applications of $- \oplus \top$ and $- \sqcup \mathbf{2}$.

$\text{cd}(\mathbf{2}) = 1$ since $\mathbf{2}$ is commutative

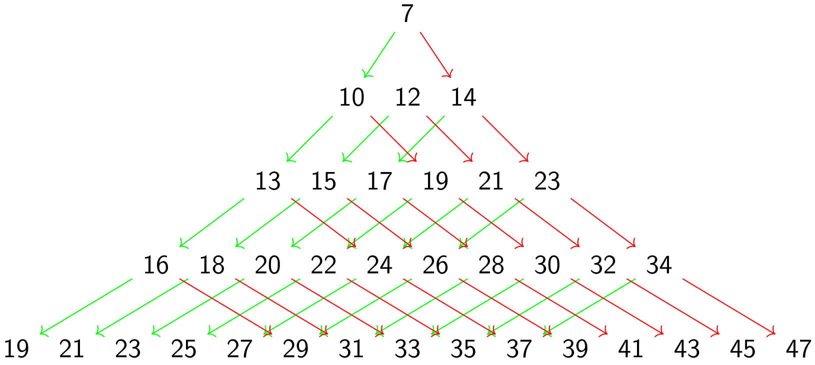
$\text{cd}(\mathbf{3}^{\mathsf{p}}) = \text{cd}(\mathbf{2} \oplus \top) = \frac{7}{9}$, which is the only value in $\mathcal{CD}(3)$.

Obtaining values in $\mathcal{CD}(4)$ and $\mathcal{CD}(5)$

$\frac{10}{16}$	$\frac{12}{16}$	$\frac{14}{16}$
$\mathbf{3}^{\mathsf{p}} \oplus \top$	$\mathbf{3}^{\mathsf{c}} \oplus \top$	$\mathbf{3}^{\mathsf{p}} \sqcup \mathbf{2}$

$\frac{13}{25}$	$\frac{15}{25}$	$\frac{17}{25}$	$\frac{19}{25}$	$\frac{21}{25}$	$\frac{23}{25}$
$(\mathbf{3}^{\mathsf{p}} \oplus \top) \oplus \top$	$(\mathbf{3}^{\mathsf{c}} \oplus \top) \oplus \top$	$(\mathbf{3}^{\mathsf{p}} \sqcup \mathbf{2}) \oplus \top$	$(\mathbf{3}^{\mathsf{p}} \oplus \top) \sqcup \mathbf{2}$	$(\mathbf{3}^{\mathsf{c}} \oplus \top) \sqcup \mathbf{2}$	$(\mathbf{3}^{\mathsf{p}} \sqcup \mathbf{2}) \sqcup \mathbf{2}$

Obtaining values in $\mathcal{CD}(n)$



Numerators appearing in $\mathcal{CD}(n)$ for $n = 3, 4, 5, 6, 7$

Green arrow $= - \oplus \top$

Red arrow $= - \sqcup \mathbf{2}$

Theorem (E., 2025)

For every $n \geq 3$, every commuting degree in $\mathcal{CD}(n)$ is achieved by an algebra of order n.

Sketch.

Induct on n.

Referring to the above diagram: at level n, there are T_{n-2}-many algebras.

- Apply $- \oplus \top$ to each algebra on level n, increasing the numerators of their commuting degrees by 3.
- Apply $- \sqcup \mathbf{2}$ to only the last $n-1$ algebras on level n, increasing the numerators of their commuting degrees by $2n+1$.

The above procedure builds $T_{n-2} + (n-1) = T_{n-1}$ algebras.

Tedious computation shows this yields every commuting degree on level $n+1$.

□

In general, each commuting degree can be obtained in several ways, with one exception: the minimum.

Define a family of algebras:

$$\mathbf{M}_3 = \mathbf{3}^{\mathrm{p}}$$
$$\mathbf{M}_n = \mathbf{M}_{n-1} \oplus \top$$

for $n > 3$.

Theorem (E., 2025)

Up to isomorphism, the algebra $\mathbf{M}_n$ is the unique BCK-algebra of order n with commuting degree $\frac{3n-2}{n^2}$, the minimum value in $\mathcal{CD}(n)$.

Theorem (E., 2025)

Every rational in $(0,1)$ is the commuting degree for some finite non-commutative BCK-algebra. That is,

$$\bigcup_{n=3}^{\infty} \mathcal{CD}(n) = \mathbb{Q} \cap (0,1)\,.$$

Sketch.

Take $\frac{p}{q} \in \mathbb{Q} \cap (0,1)$.

Set $n = 2q$ and $k = 2q(q-p)$.

Check that $k \in \{1, 2, \ldots, T_{n-2}\}$.

Then $\frac{p}{q} = \frac{n^2-2k}{n^2} \in \mathcal{CD}(n)$.

□

Example

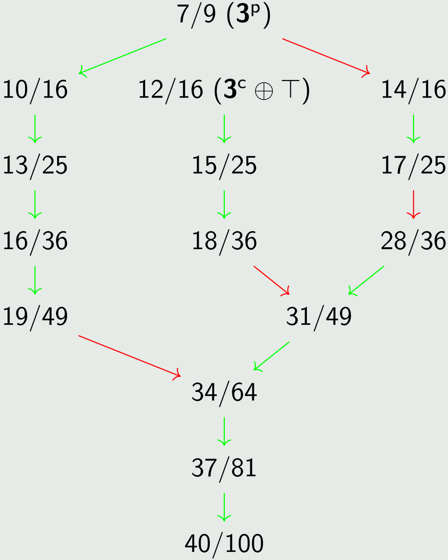
Suppose we want a BCK-algebra with commuting degree $\frac{2}{5}$.

We need to consider algebras of order $n = 2q = 10$.

From the proof of earlier theorem, there are three algebras with this commuting degree:

$$((((((\mathbf{3}^{\mathsf{p}} \oplus \top) \oplus \top) \oplus \top) \oplus \top) \sqcup \mathbf{2}) \oplus \top) \oplus \top$$
$$((((((\mathbf{3}^{\mathsf{c}} \oplus \top) \oplus \top) \oplus \top) \sqcup \mathbf{2}) \oplus \top) \oplus \top) \oplus \top$$
$$((((((\mathbf{3}^{\mathsf{p}} \sqcup \mathbf{2}) \oplus \top) \sqcup \mathbf{2}) \oplus \top) \oplus \top) \oplus \top) \oplus \top$$

Example


7/9 (3^p)
10/16
12/16 (3^c ⊕ ⊤)
14/16
13/25
15/25
17/25
16/36
18/36
28/36
19/49
31/49
34/64
37/81
40/100

Closing questions

Questions

- Can commuting degree (or satisfiability degree of other equations) tell us structural information about the algebra?
- Which equations in one variable have finite satisfiability degree?
- Are there any equations in two variables with finite satisfiability degree?
- Can we generalize to infinite algebras?

Thanks!

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