RUTGERS
THE STATE UNIVERSITY
OF NEW JERSEY

Groups, orders, & classification

joint with

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Aknowledgements

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The Continuum Hypothesis (CH)

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- Cohen's method of forcing showed that $\neg$**CH** cannot be refuted. (1963)

Whether **CH** is solved or not is an endless dispute. Settling **CH** is still one of the main motivation for current research on the foundation of mathematics. (E.g., Woodin's **V= Ultimate L** program.)

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Even though it was verified in some special cases, Vaught's conjecture remains open.

Classification

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As the formulation of Vaught's conjecture might suggest, this can be done simply by **counting equivalence classes**.

Theorem (approx. 300 B.C.)

There are exactly five convex regular polyhedra (i.e. Platonic solids).

Figure: Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron.

Classifications beyond Vaught's conjecture

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Moreover, for all $a, b \in \mathbb{N}$ there exists an o-minimal theory T such that T has exactly $6^a 3^b$ countable models.

Definition

Let E be an equivalence relation on X. A **complete classification** for E is a map $c\colon X \to I$ such that for any $x, y \in X$,

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Example

Countable Abelian reduced p-groups are completely classified by their Ulm invariants.

The descriptive set theory view

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By identifying $\mathcal{M}$ with $(R_i^{\mathcal{M}} \mid i \in I)$ we can define the **standard Borel space of (countably infinite) enumerated $\mathcal{L}$-structures** as

$$X_{\mathcal{L}} = \{\mathcal{M} \mid \mathcal{M} \text{ is an enumerated } \mathcal{L}\text{-structure}\} = \prod_{i \in I} 2^{\mathbb{N}^{a_i}}.$$

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Theorem (Lopez-Escobar)

A subset $X \subseteq X_{\mathcal{L}}$ is Borel (in which case it admits a standard Borel structure itself) if and only if $X = X_\varphi$ for some $\mathcal{L}_{\omega_1\omega}$ sentence φ.

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Church's thesis for real mathematics

Borel = explicit

Back to classifications (cont'd)

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- E is **Borel reducible** to F (in symbols $E \leq_B F$) if and only if if there exists a Borel map $f\colon X \to Y$ such that

  $$x \mathrel{E} y \iff f(x) \mathrel{F} f(y).$$

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- $E \sim_B F$ if and only if $E \leq_B F$ and $F \leq_B E$.

Definition (Friedman–Stanley 1989)

Let E be an equivalence relation on a standard Borel space X.
For $x = (x_i : i \in \mathbb{N})$ and $y = (y_i : i \in \mathbb{N})$ in $X^{\mathbb{N}}$ let

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1. The first jump $=_{\mathbb{R}}^+$ is defined on $\mathbb{R}^{\mathbb{N}}$ so that the map

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2. The second jump $=_{\mathbb{R}}^{++}$ is defined on $(\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ and admits a complete classification by **hereditarily countable set** in $\mathcal{P}(\mathcal{P}(\mathbb{R}))$.

The Friedman-Stanley jump Hierarchy

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Proposition

- *For countable $\alpha < \beta$, we have $=_{\mathbb{R}}^{\alpha+} <_B =_{\mathbb{R}}^{\beta+}$*
- *Every Borel isomorphism relation is Borel reducible to $=^{\alpha+}$ for some $\alpha < \omega_1$.*

Theorem (Rast–Sahota '17)

Let T be a complete o-minimal theory in a countable language.

1. $\cong_T \leq_B =_{\mathbb{R}}$.
2. $\cong_T \sim_B =^+_{\mathbb{R}}$.
3. $\cong_T$ *is not Borel (in fact,* $\cong_T$ *is maximal among isomorphisms).*

Ordered divisible abelian groups

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Given a linear order $\mathbf{L} = (L, <_L)$ consider the group $\mathbf{G_L} = (G_L, <_{lex})$ where

- $G_L = \{f : L \to \mathbb{Q} \mid \text{supp(f) is finite}\}$
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For $x, y \in \mathbf{G_L}$ we define

$$x \preceq y \iff \exists n \in \mathbb{Z} \setminus \{0\}\ (x \leq_{lex} ny)$$
$$x \approx y \iff x \preceq y \text{ and } y \preceq x$$

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The $\approx$-classes are maximal Archimedean subgroups. From the quotient set $\mathbf{G_L}/\approx$ it is possible to recover the order $\mathbf{L}$, so that

$$\mathbf{G_L} \cong_{\mathsf{ODAG}} \mathbf{G_K} \implies \mathbf{L} \cong_{\mathsf{LO}} \mathbf{K}.$$

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What is the Borel complexity of the isomorphism relation $\cong_{\mathsf{ArGp}}$ for countable ordered Archimedean groups?

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- *G has an Archimedean order.*
- *G acts freely on $\mathbb{R}$ by orientation preserving.*
- *G is isomorphic to a subgroup of $(\mathbb{R}, +)$ equipped with the natural ordering on $\mathbb{R}$.*

Hion's lemma

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Theorem (C.-Marker–Motto Ros–Shani 2023)

There is a continuous map $X_{\mathsf{ArGp}} \to \mathcal{A}$ showing $\cong_{\mathsf{ArGp}} \leq_B \cong_{\mathcal{A}}$.

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Suppose that A and B are two subgroups of $\mathbb{R}$ and $h\colon A \to B$ is an order preserving homomorphism. Then, there exists $\lambda \in \mathbb{R}^+$ such that $h(a) = \lambda a$, for every $a \in A$.

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Lemma (Hion 54')

Suppose that A and B are two subgroups of $\mathbb{R}$ and $h\colon A \to B$ is an order preserving homomorphism. Then, there exists $\lambda \in \mathbb{R}^+$ such that $h(a) = \lambda a$, for every $a \in A$.
In fact, such λ is computed as the ratio $\frac{h(a)}{a}$, for any nonzero $a \in A$.

If G is a nontrivial subgroup of $\mathbb{R}$ let

$$A_G \coloneqq \left\{ \underbrace{\left\{ \frac{g}{r} : g \in G \right\}}_{G/r} : r \in G \setminus \{0\} \right\}.$$

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Proposition

Let G and H be non-trivial subgroups of $\mathbb{R}$. Then

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Therefore, $\cong_{\mathsf{ArGp}} \leq_B =^{++}_{\mathbb{R}}$.

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Note that for $r \neq s$, the sets G/r and G/s are not at odds with each other.

The case for a finer hierarchy

(Hjorth–Kechris–Louveau 1998) defined a refinement of the Friedman–Satanley hierarchy. In particular,

$$=^{+} <_B \cong^*_{3,0} <_B \cong^*_{3,1} <_B =^{++}$$

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An invariant for $\cong^*_{3,1}$ is a hereditarily countable set $A \in \mathcal{P}_2(\mathbb{R})$ (i.e., a $=^{++}$-invariant) together with

- a ternary relation $R \subseteq A \times A \times \mathbb{R}$, definable from A, such that given any $a \in A$, $R(a, -, -)$ is an injective function from A to $\mathbb{R}$.

Main theorem

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We cannot use countable sets of reals to classify $\cong_{\mathsf{ArGp}}$.

A **circular order** on a set X is defined by a **cyclic orientation cocycle**, i.e., a function $c\colon X^3 \to \{\pm 1, 0\}$ satisfying:

1. $c^{-1}(0) = \Delta(X)$, where
   $\Delta(X) \coloneqq \{(x_1, x_2, x_3) \in X^3 : x_i = x_j, \text{ for some } i \neq j\}$,
2. $c(x_2, x_3, x_4) - c(x_1, x_3, x_4) + c(x_1, x_2, x_4) - c(x_1, x_2, x_3) = 0$ for all $x_1, x_2, x_3, x_4 \in X$.

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Definition

A group G is **circularly orderable** if it admits a circular order c which is left-invariant in the sense that $c(g_1, g_2, g_3) = c(hg_1, hg_2, hg_3)$ for all $g_1, g_2, g_3, h \in G$.

Definition

A circularly ordered group (G, c) is said to be **Archimedean** if there are no elements $g, h \in G$ such that $c(1_G, g^n, h) = 1$ for all $n \geq 1$.

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A well-known example of Archimedean circularly ordered group is S^1 with the obvious circular order.

There is a classical functorial construction that starting from a circular order group (G, c) produces a corresponding ordered group $(\widetilde{G}, <)$ called **central extension**. (Želeva)

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This can be used to show that

$$\cong_{\mathsf{ArCO}} \leq_B \cong_{\mathsf{ArGp}}.$$

We also showed the following:

Theorem (C.–Marker–Motto Ros–Shani 2023)

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No Borel converse to central extensions

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Theorem (C.–Marker–Motto Ros–Shani 2023)

$\cong_{\mathsf{ArCO}} \leq_B =^+_{\mathbb{R}}$.

There is NO Borel (explicit) converse to the central extension functor because $\cong_{\mathsf{ArCO}} <_B \cong_{\mathsf{ArGp}}$.

Enter set theory

The proof of $\cong_{\mathsf{ArGp}} \not\leq_B \cong^*_{3,0}$ is highly nonelementary and requires the analysis of the complete invariants of $\cong_{\mathsf{ArGp}}$ and $\cong^*_{3,0}$ in choiceless models of set theory.

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Definition

Suppose A is a set in some generic extension of V . Let $V(A)$ be the **minimal transitive model** of **ZF** containing V and A, denoted by $V(A)$.

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Definition

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For any set X, there is some formula ψ, parameters $\bar{a} \in tc(A)$ and $v \in V$ such that X is the unique set satisfying $\psi(X, A, \bar{a}, v)$.

We use the following powerful technique:

Theorem (essentially Shani 2018)

Suppose E is a Borel equivalence relation on a standard Borel space X, and $x \mapsto A_x$ is an absolute classification of E by hereditarily countable sets.

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Suppose E is a Borel equivalence relation on a standard Borel space X, and $x \mapsto A_x$ is an absolute classification of E by hereditarily countable sets. Let x be an element of X in some generic extension of V. If $E \leq_B \cong^*_{3,0}$, then there is a set of sets of reals $B \in V(A_x)$ so that:

- *B is definable from A_x and parameters in V,*
- *B is countable in $V(A_x)$,*
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Proof of the main theorem: Over the Cohen model, we force the existence of a generic subgroup G of $\mathbb{R}$ so that every set of reals $B \in V(A_G)$ which is definable from A_G and parameters in V alone, we have $V(B) \neq V(A_G)$.

Thank You!