The finite condensation and the lattice of ordinals of finite degree

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Recall . . .

- We have the left rectangular band $R^{\mathbf{ON}}$ acting on End($\mathbf{ON}$) by the left and right actions $1 \mapsto \phi^R_{\text{left}(1)}$ and $1 \mapsto \phi^R_{\text{left}(\omega)}$ where $\phi^R_{\text{left}(1)}(\alpha) = \partial_F(\alpha) = 1 \cdot_F \alpha = \text{o.t.}(\alpha/\sim_F)$ and $\phi^R_{\text{left}(\omega)}(\alpha) = \omega \cdot_F \alpha$.
- Recall that we have the map ∂_F acting on $\omega[\omega]^{\omega}_{CNF}$, the Cantor normal forms of ordinals of finite degree:
- Suppose α is an ordinal of finite degree with Cantor normal form $\Phi(\alpha) = a_n\omega^n + a_{n-1}\omega^{n-1} + \cdots + a_1\omega + a_0$, with $n > 0$. Then (by abuse of notation, writing $\partial_F(\alpha)$ for $\partial_F(\Phi(\alpha))$):

$$\partial_F(\alpha) = a_n\omega^{n-1} + a_{n-1}\omega^{n-2} + \cdots + a_1 + c_\alpha$$

where $c_\alpha = 0$ if $a_0 = 0$, and $c_\alpha = 1$ if $a_0 \neq 0$.

Left multiplication by ω mod finite

Theorem

Let $\alpha \in \omega[\omega]^{\omega}_{CNF}$ be an ordinal of finite degree with Cantor normal form $\alpha = a_n\omega^n + a_{n-1}\omega^{n-1} + \cdots + a_1\omega + a_0$. Then $\phi_I^F(\omega)(\alpha) = \omega^n$; that is, $\phi_I^F(\omega)(\alpha) = \omega^{\deg(\alpha)}$.

Sketch of Proof:

- For any $n \in \omega$, ${\omega^{n+1}}/{\sim_F} \cong \omega^n$.
- If $m < n < \omega$, then ω^n begins with ω-many copies of ω^m, and there is a well-ordered set D such that as a linear order, $\omega^n \cong \omega\omega^m + D$.
- If $m < n < \omega$, then $\omega^m + \omega^n \cong \omega^n$.

Left multiplication by ω mod finite

- If $m < n < \omega$ and $a, b \in \omega$ with $a, b \neq 0$, then $a\omega^m + b\omega^n \cong b\omega^n$.
- Let α be an ordinal of finite degree with Cantor normal form $\alpha = a_n\omega^n + \cdots + a_1\omega + a_0$. Then for each $k \in \omega$, $k\alpha$ is an initial segment of $(k+1)a_n\omega^n$.
- Suppose α is an ordinal of finite degree and $\alpha \cong a_n\omega^n + \cdots + a_1\omega + a_0$ in Cantor normal form. Then $\omega\alpha \cong \omega\omega^n \cong \omega^{n+1}$.

Left multiplication by ω mod finite

- Now: Suppose $\alpha = a_n\omega^n + a_{n-1}\omega^{n-1} + \cdots + a_1\omega + a_0$ is an ordinal of degree n in Cantor normal form, for some $n \in \omega$. By an earlier lemma, $\omega\alpha \cong \omega^{n+1}$. Then by another earlier lemma,

$$\phi_I^F(\omega)(\alpha) = \omega \cdot_F \alpha = \text{o.t.}({}^{\omega\alpha}/_{\sim_F}) \cong {}^{\omega^{n+1}}/_{\sim_F} \cong \omega^n.$$

$\square$

Monic monomials

- So, the map $\phi^F_{\mathrm{left}}(\omega)$ maps any ordinal $\alpha \in \omega[\omega]^{\omega}_{CNF}$ to the "monic monomial" $\omega^{\mathrm{deg}(\alpha)}$; the leading coefficient and the lower terms are lost.
- That is,

$$\phi^F_{\mathrm{left}}(\omega) : \omega[\omega]^{\omega}_{CNF} \to \mathrm{MMonomials}(\omega[\omega]^{\omega}_{CNF}).$$

- Noting that $\mathrm{MMonomials}(\omega[\omega]^{\omega}_{\mathrm{CNF}})$ can be identified with ω, we have that $\phi^F_{\mathrm{left}}(\omega)$ can be identified with the degree map on $\omega[\omega]^{\omega}_{CNF}$.

A remark

We have only defined ∂_F on Cantor normal forms, and the expression $\alpha + \beta = a_n\omega^n + \cdots + a_0 + b^m + \cdots + b_0$ need not be in Cantor normal form. For ease of notation, we will write "$\partial_F(p\alpha)$" and "$\partial_F(\alpha + \beta)$" for $\partial_F(\Phi(p\alpha))$ and $\partial_F(\Phi(\alpha + \beta))$ respectively; it will be understood that the argument of ∂_F in such expressions is put into Cantor normal form before the finite condensation derivative is taken. For similar reasons, we will write "$\partial_F(\alpha) + \partial_F(\beta)$" to mean $\Phi(\partial_F(\alpha) + \partial_F(\beta))$.

Pseudo-linearity

Proposition (B–S)

Let $\alpha = a_n\omega^n + \cdots + a_0$ *be an ordinal of finite degree, and let* $p \in \omega$ *with* $p \geq 1$*. Then* $p\alpha \cong (p-1)a_n\omega^n + \alpha \cong pa_n\omega^n + a_{n-1}\omega^{n-1} + \cdots + a_0$*.*

Proposition (B–S)

Let $\alpha = a_n\omega^n + \cdots + a_0$ *be an ordinal of finite degree* $n \geq 1$*, and let* $p \in \omega$ *with* $p > 0$*.*

1. *If* α *has degree* $n \geq 2$*, then* $\partial_F(p\alpha) \cong p\partial_F(\alpha) \cong (p-1)a_n\omega^{n-1} + \partial_F(\alpha)$*.*
2. *If* α *has degree* 1*, then* $\partial_F(p\alpha) \cong (p-1)a_n\omega^{n-1} + \partial_F(\alpha)$*, and* $p\partial_F(\alpha)$ *differs from* $(p-1)a_n\omega^{n-1} + \partial_F(\alpha)$ *by at most a constant.*

Pseudo-linearity

Proposition (B–S)

Let α and β be nonzero ordinals of finite degree in Cantor normal form. Then we have $\partial_F(\alpha + \beta) \cong \partial_F(\alpha) + \partial_F(\beta)$ in the following cases:

1. *α is a limit ordinal; or*
2. *α is a successor ordinal and $\deg \beta \geq 2$.*

In all other cases, we have $\partial_F(\alpha + \beta) + 1 \cong \partial_F(\alpha) + \partial_F(\beta)$.

Pseudo-linearity

Corollary (B–S)

Suppose, for $1 \leq i \leq t$, that α_i is a limit ordinal of finite degree in Cantor normal form. Then

$$\partial_F \left(\sum_{i=1}^{t} \alpha_i \right) \cong \sum_{i=1}^{t} \partial_F(\alpha_i).$$

□

Pseudo-linearity

Theorem (B–S)

Let $\alpha = a_n\omega^n + \cdots + a_1\omega$ *and* $\beta = b_m\omega^m + \cdots + b_1\omega$ *be limit ordinals of finite degree at least 2 in Cantor normal form, and let* $p, q \in \omega$ *with* $p, q > 0$. *Then* $\partial_F(p\alpha + q\beta) = p\partial_F(\alpha) + q\partial_F(\beta)$.
Moreover, we have the following expressions for $\partial_F(p\alpha + q\beta)$:

1. *If* $\deg(\alpha) < \deg(\beta)$, *then* $\partial_F(p\alpha + q\beta) \cong qb_m\omega^{m-1} + b_{m-1}\omega^{m-2} + \cdots + b_2\omega + b_1$;
2. *If* $\deg(\alpha) = \deg(\beta)$, *then*
$$\partial_F(p\alpha + q\beta) \cong (pa_n + qb_n)\omega^{n-1} + b_{n-1}\omega^{n-2} + \cdots + b_2\omega + b_1;$$
3. *If* $\deg(\alpha) > \deg(\beta)$, *then* $\partial_F(p\alpha + q\beta)$
$$\cong pa_n\omega^{n-1} + a_{n-1}\omega^{n-2} + \cdots$$
$$\cdots + a_{m+1}\omega^m + (a_m + qb_m)\omega^{m-1} + b_{m-1}\omega^{m-2} + \cdots + b_1.$$

The dual map $\partial_F^\star$

- Denote $\partial_F^\star(\alpha) := \{\beta \in \omega[\omega]^\omega_{CNF} : \partial_F(\beta) = \alpha\}$.

Proposition

Suppose $\alpha = a_n\omega^n + \cdots + a_0$ is a nonzero ordinal of finite degree.

1. *If $a_0 = 0$, then $\partial_F^\star(\alpha) = \{a_n\omega^{n+1} + \cdots + a_1\omega^2\}$.*
2. *If $a_0 = 1$, then $\partial_F^\star(\alpha)$ consists of $a_n\omega^{n+1} + \cdots + a_1\omega^2 + \omega$ along with all ordinals of the form $a_n\omega^{n+1} + \cdots + a_1\omega^2 + j$ for $j \in \omega$, $j > 0$.*
3. *If $a_0 > 1$, then $\partial_F^\star(\alpha)$ consists of $a_n\omega^{n+1} + \cdots + a_1\omega^2 + a_0\omega$ along with all ordinals of the form $a_n\omega^{n+1} + \cdots + a_1\omega^2 + (a_0 - 1)\omega + j$ for $j \in \omega$, $j > 0$.*

$\square$

The lattice of ordinals of finite degree

- Ordinals of the form $a\omega^n$ can be identified with points in the 2-dimensional lattice $\mathbb{N} \times \mathbb{N}$, where the ordinal $a\omega^n$ is associated with (a, n).
- Acting on $a\omega^n$ by the map ∂_F corresponds to moving the point (a, n) down one unit in the lattice.
- Acting on $a\omega^n$ by the map $\partial_F^\star$ corresponds to moving the point (a, n) up one unit in the lattice.
- The map $\phi_{\mathrm{left}}^F(\omega)$ moves the point (a, n) corresponding to $a\omega^n$ to the point $(1, n)$.

Future work

We plan to address the following questions.

1. Which of the properties of ∂_F acting on ordinals of finite degree remain true if we consider ∂_F acting on countable ordinals of degree at least ω?
2. If the finite condensation is replaced by another condensation, and if we define a multiplication of linear orders based on that condensation, what algebraic structures arise?
3. Given a condensation $\sim$, can we characterize the set of linear orders L such that $L/\sim \cong 1$?

Thank-you!

Thank you to the organizers of BLAST for the opportunity to speak today.

A couple of references

- Standard reference on linear orders:
  J. Rosenstein, *Linear Orderings*, Academic Press, 1982.
- Our paper associated with these slides:
  J. Brown and R. Suárez, Algebraic structures arising from the finite condensation on linear orders. (submitted; current version [v3] available on arXiv after 27 May 2025)