A construction of the assembly of a frame

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Outline

1. Overview
2. The bounded meet semilattice D of differences
3. Frame congruences
4. Sites and coverages
5. The assembly is free over D
6. The assembly is an essential extension of L

Overview

The assembly of a frame L has several important manifestations.

- It is the congruence lattice $\operatorname{con} L$ of the frame.
- It is the lattice NL of nuclei on the frame.
- It is an extension $L \to NL$ which can characterized as the result of freely complementing the elements of L.

The assembly has a number of significant properties.

- NL is an essential extension of L, and its skeleton $L \to NL \to NL^{**}$ is the essential completion of both.
- NL is ultranormal: any disjoint pair of sublocales can be separated by a complemented sublocale.
- NL is ultraparacompact: every cover has a pairwise disjoint refinement.

In this talk we will construct the assembly of L using Johnstone's method of sites and coverages, starting with the meet semilattice of differences of elements of L. The construction permits succinct proofs of its properties, on which we shall comment as time permits.

The bounded meet semilattice D of differences

Definition

We denote the order relation on L by

$$LEQ \equiv \left\{ (a, b) \in L^2 : a \leq b \right\},$$

and for $(a, b) \in LEQ$ we denote its interval frame by

$$[a, b] \equiv \{ c : a \leq c \leq b \}$$

and its projection homomorphism by

$$p_a^b : L \to [a, b] = \left(c \mapsto a \vee (c \wedge b) = (c \vee a) \wedge b \right) \quad c \in L.$$

The bounded meet semilattice D of differences

Definition

We preorder LEQ by declaring, for elements $(a, b), (c, d) \in LEQ$, that

$$(a, b) \ll (c, d) \text{ if } c \wedge b \leq a \text{ and } d \vee a \geq b.$$

We say that (a, b) *distinguishes* (c, d). We denote the preorder equivalence class of an element $(a, b) \in LEQ$ by

$$\langle a, b \rangle \equiv \left\{ (c, d) : (a, b) \ll (c, d) \ll (a, b) \right\};$$

use of the notation $\langle a, b \rangle$ will presume that $a \leq b$. We refer to

$$D \equiv \left\{ \langle a, b \rangle : (a, b) \in LEQ \right\}$$

as the *meet semilattice of differences of* L. We denote its members by lower case letters towards the end of the Latin alphabet, e.g. x, y, z etc.

The bounded meet semilattice D of differences

Lemma

D is a bounded meet semilattice in which $\top = \langle \bot, \top \rangle$, $\bot = \langle a, a \rangle$ for any $a \in L$, and

$$\langle a, b \rangle \wedge \langle c, d \rangle = \langle p_c^d(a), p_c^d(b) \rangle = \langle p_a^b(c), p_a^b(d) \rangle$$

for $\langle a, b \rangle, \langle c, d \rangle \in D$.

Corollary

For $\langle a, b \rangle, \langle c, d \rangle \in D$,

$$\langle a, b \rangle \wedge \langle c, d \rangle = \bot \iff p_c^d(a) = p_c^d(b) \iff p_a^b(c) = p_a^b(d)$$

D is closely connected to frame congruences

Keep in mind that a frame congruence contains a pair (a, b) if and only if it contains the pair $(a \wedge b, a \vee b)$.

Lemma

For any pair $(a, b) \in LEQ$*, the finest (smallest) frame congruence which identifies a with b, designated* ϕ_x*, is*

$$\{(c, d) : (c \wedge d, c \vee d) \ll (a, b)\}$$

We denote this congruence ϕ_x *for* $x \equiv \langle a, b \rangle$*.*

conL contains a copy of D

Theorem

The map $m \colon D \to \operatorname{con} L = (x \mapsto \phi_x)$ *is an injective meet semilattice homomorphism.*

The coverage on D

We are interested in certain downsets of D.

Definition

We call a downset $U \subseteq D$ an *ideal* if it is

- *transitive*, i.e., $\langle a, b\rangle, \langle b, c\rangle \in U$ implies $\langle a, c\rangle \in U$, and
- *closed under the meets in L*, i.e., $\langle a_i, b_i\rangle \in U$ implies $\langle a_1 \wedge a_2, b_1 \wedge b_2\rangle \in U$.

We shall say that an ideal U covers an element $x = \langle a, b\rangle \in D$, and write $U \sqsupseteq x$, if

$$b = \bigvee \{ c \in [a, b] : \langle a, c\rangle \in U \}$$

Lemma

The relation $\sqsupseteq$ serves as a coverage on D. That is, for any ideal $U \subseteq D$ and any elements $x, y \in D$ we have

- *$x \in U$ implies $U \sqsupseteq x$, and*
- *$U \sqsupseteq x \gg y$ implies $U \sqsupseteq y$.*

The frame M of coverage ideals

Definition

A *coverage ideal* on D is an ideal $U \subseteq D$ which is closed under the coverage, i.e., $U \sqsupseteq x$ implies $x \in U$ for all $x \in D$. We denote the frame of coverage ideals by M.

What is this frame M of coverage ideals?

M is isomorphic to NL

Theorem

For any nucleus $j \in NL$, the downset

$$V_j \equiv \{ \langle a, b \rangle \in D : b \leq j(a) \}$$

is a coverage ideal of D.
For any coverage ideal $U \subseteq D$, the map

$$k_U : L \to L = \left(a \mapsto \bigvee \{ b : \langle a, b \rangle \in U \}\right)$$

is a nucleus on L.
The maps

$$k : M \to NL = (U \mapsto k_U) \quad \text{and} \quad v : NL \to M = (j \mapsto V_j)$$

are inverse order-preserving bijections.

The assembly is free over D

A consequence of the theorem is that the assembly is free over D. To explain what is meant here, first note that M contains a canonical copy of D.

Lemma

For each $x \in D$, $\downarrow x \downarrow_D$ is a coverage ideal. In fact, the map

$$l \colon D \to M = (x \mapsto \downarrow x \downarrow_D), \quad x \in D$$

is an injective bounded meet semilattice homomorphism which makes the diagram commute.

$$\begin{array}{ccc} D & \xrightarrow{m} & \operatorname{con} L \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle n} \\ M & \underset{v}{\overset{k}{\rightleftarrows}} & NL \end{array}$$

Let $D' \equiv l(D)$ be the copy of D in M. D' in inherits the coverage $\sqsupseteq$ on D, and we conflate D' with D.

M is free over D

Theorem

M is free over D in the following sense.

- *M contains a copy of D as a join generating bounded meet sub-semilattice, and*
- *for any frame K and any bounded meet semilattice homomorphism $f : D \to K$ which transforms coverings to joins, f lifts to a unique frame homomorphism $M \to K$.*

The assembly is an essential extension of L

Lemma

A frame injection $f\colon L \to K$ is an essential extension if and only if every pair from K is distinguished by (the image of a) pair from L. In symbols, for every $c < d$ in K there must exist $a < b$ in L such that $\langle f(a), f(b)\rangle \ll \langle c, d\rangle$.

Theorem

The map $\kappa\colon L \to \operatorname{con} L = (a \mapsto \kappa_a)$ is an essential extension. In fact, the map

$$L \xrightarrow{\kappa} \operatorname{con} L \xrightarrow{n} NL \xrightarrow{s} NL^{**},$$

where $s\colon NL \to NL^{**}$ is the skeleton map on NL, is the essential completion of both L and NL.

Thank you.