What Are Coz-inclusions?

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Our Categories

Loc$=$ **Frm**^{op}
Locales

- Locales
- Localic maps

Frm
Frames

- Frames Complete Lattice +
  $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$
- Frame homomorphisms (preserve $\bigvee$ and $\wedge$)

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Topological Spaces

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- Continuous maps

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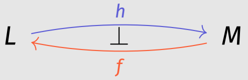

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$\Omega(-)$

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  $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$

  $\Omega(x) = \tau$

- Frame homomorphisms (preserve $\bigvee$ and $\wedge$)

  $\Omega(f) = f^{-1} \colon \Omega(Y) \to \Omega(X)$

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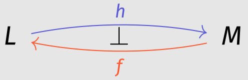
  (X, τ)

- Continuous maps

  $f : X \to Y$



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Our subobjects? Extremal monomorphisms in **Loc**

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Sublocales ($\neq$ subframes) | **Quotients** | **Subspaces**

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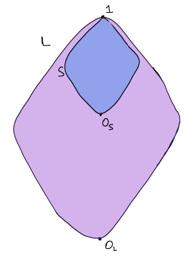

Quotients

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A *subspace* of a topological space X is a subset $S \subseteq X$ with the subset topology.

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$\mathcal{S}(L)$, $\subseteq$, $\bigwedge$, $\bigsqcup$, $0 = \{1\}$, L

Coframe ($\mathcal{S}(L)^{op}$ is a frame)

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$\mathcal{P}(X)$, $\subseteq$, $\bigcap$, $\bigcup$, $\emptyset$, X, $\_^c$

Complete atomic boolean algebra.

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S sublocale of L $\rightsquigarrow$ $L \xrightarrow{q} S$, $\quad x \longmapsto \bigvee \{s \in S \mid x \leq s\}$

$h_*[M]$ is a sublocale of L $\rightsquigarrow$ $L \underset{h_*}{\overset{h}{\rightleftarrows}} M$

$\prec\!\!\prec$ and Complete Regularity

- **Rather below relation**: for $a, b \in L$

$$a \prec b \quad \equiv \quad \exists\, c \in L,\ a \wedge c = 0 \text{ and } c \vee b = 1$$

Spatial case: X a topological space and $U, V \in \Omega(X)$, $U \prec V$ iff $\overline{U} \subseteq V$

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- **Completely below relation**: Largest interpolative relation contained in $\prec$. Equiv. for $a, b \in L$

$$a \prec\!\!\prec b \quad \equiv \quad \text{There is } \{a_q\}_{q \in \mathbb{Q} \cap [0,1]} \subseteq L \text{ such that}$$
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Topological analogue

A space X is *completely regular* (or Tychonoff) if it is Hausdorff and whenever F is a closed set and x a point there is a continuous $\varphi \colon X \to \mathbb{R}$ such that $F \subseteq \varphi^{-1}[\{0\}]$ and $x \in \varphi^{-1}[\{1\}]$.

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- A frame L is **completely regular** if $a = \bigvee \{b \in L \mid b \prec\!\!\prec a\}$ for every $a \in L$.
- Every sublocale of a completely regular locale is completely regular.

The Cozero Elements of a Frame

- A **cozero** element is an element $c \in L$ such that $c = \bigvee_{n \in \mathbb{N}} c_n$ where $c_n \prec\!\!\prec c$.

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Let (X, τ) be a topological space.
A subset $Z \subseteq X$ is a *zero set* if $Z = \varphi^{-1}[\{0\}]$ for some continuous $\varphi\colon X \to \mathbb{R}$.
A subset $C \subseteq X$ is a *cozero set* if $C = \varphi^{-1}[\mathbb{R} \setminus \{0\}]$ for some continuous $\varphi\colon X \to \mathbb{R}$. Equivalently, if $C = X \setminus Z$ for some zero set Z of X.

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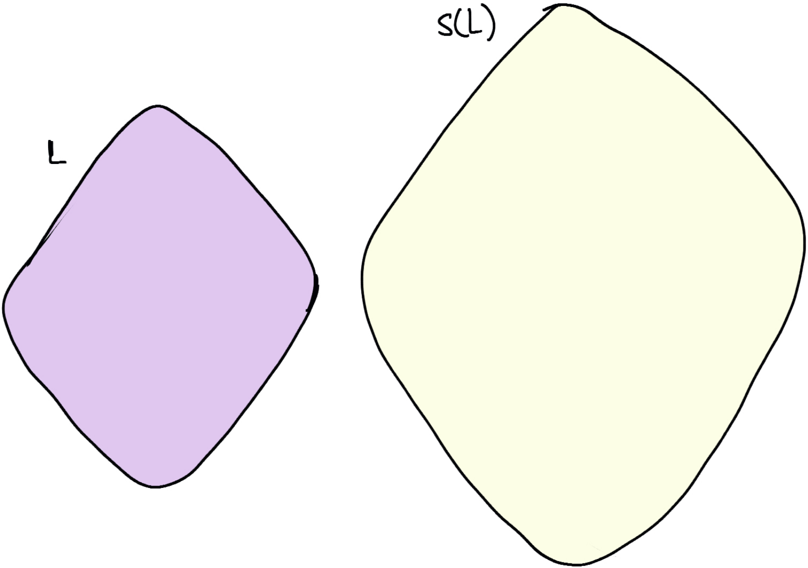
One can define zero sublocales and cozero sublocales, and they behave "almost" the same as zero and cozero sets. For each cozero element we get a cozero sublocale and a zero sublocale.

Cozero Elements and Cozero and Zero Sublocales



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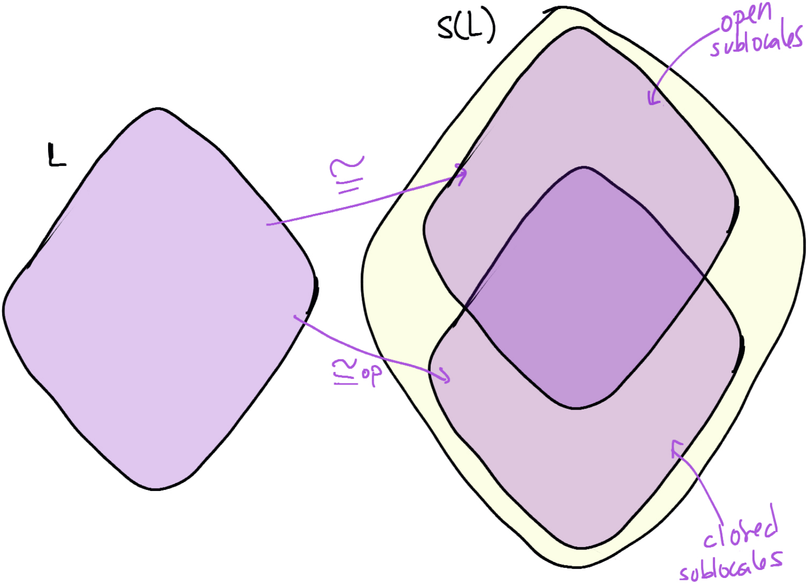

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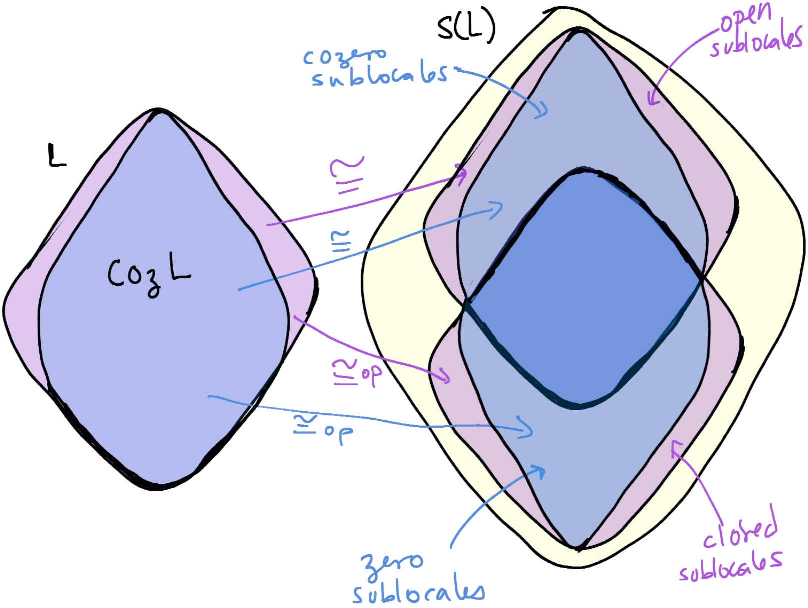

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- Recall that sublocales of completely regular frames are completely regular.
- If S is a sublocale of L, then $\text{Coz}\, S$ is a subσ-frame of S that $\bigvee$-generates S. (joins in S)

Coz S **vs.** Coz L

If S is a sublocale of L and we think of Coz S and Coz L ...

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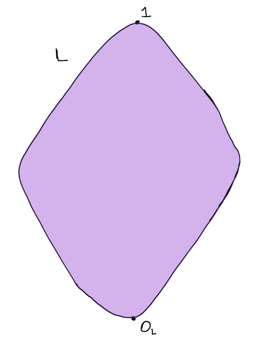

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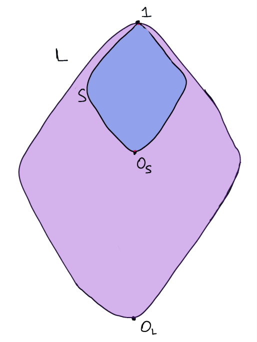

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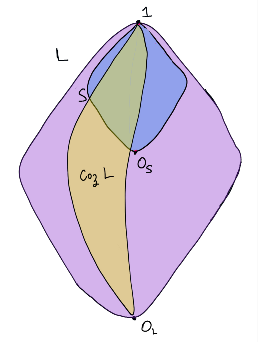

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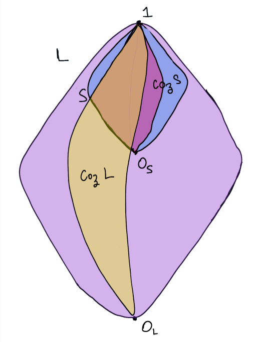

z-Embeddings

Let $S \subseteq L$ be a sublocale of L and $q\colon L \twoheadrightarrow S$ its corresponding frame quotient. If we take the restriction of q to their cozero parts (frame homomorphisms preserve cozero elements) we obtain

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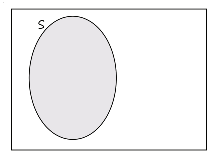

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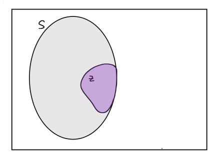

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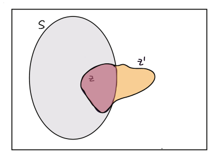

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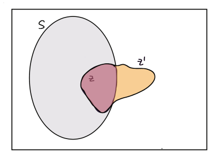

Every zero sublocale Z of S can be seen as the intersection $Z' \cap S$ for some zero sublocale Z' of L.

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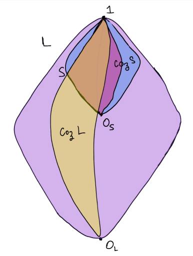

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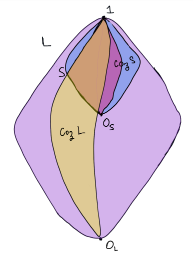

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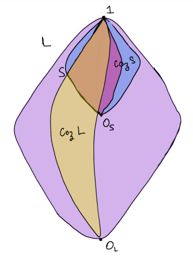
For instance, when does $\mathrm{Coz}\, S \subseteq \mathrm{Coz}\, L$ **?**



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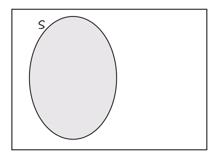

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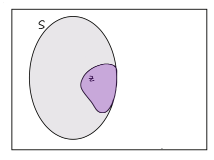

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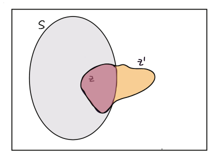

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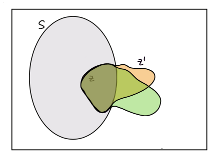

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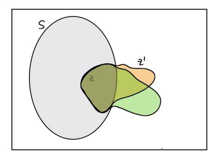

Given a zero sublocale Z of S, is there a smallest zero sublocale Z' of L such that $Z = Z' \cap S$? If there is, it has to be the closure $\overline{Z}^L$ which means the closure is a zero sublocale.

$\overline{Z}^L = \bigcap\{Z' \text{ zero sublocale } \mid Z = Z' \cap S\}$

S is coz-included if for every zero sublocale Z of S its closure $\overline{Z}^L$ is a zero sublocale of L.

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  But...under what conditions can we get the implication?

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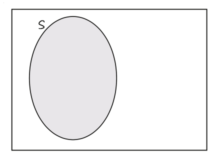

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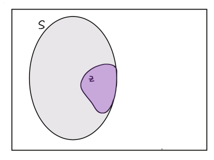

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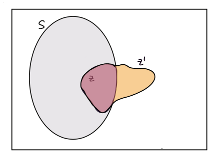
Topological intuition (Localic interpretation)



A zero sublocale Z of S that can be seen as the intersection $Z' \cap S$ for some zero sublocale Z' of L is a bueno zero sublocale.

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z-embedded (coz-onto) one-point compactification...

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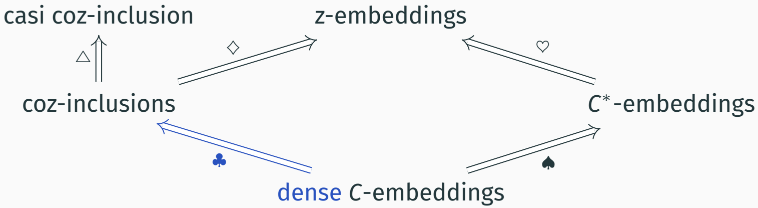
- Thus, S is casi coz-included, but it is not z-embedded.

Some Implications Among Types of Embeddings



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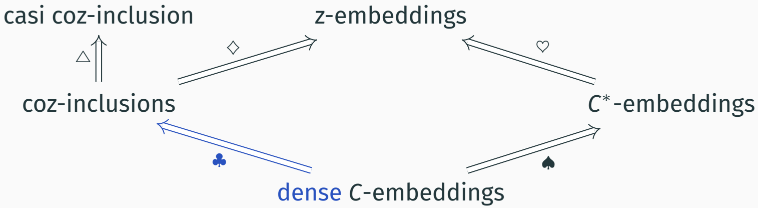

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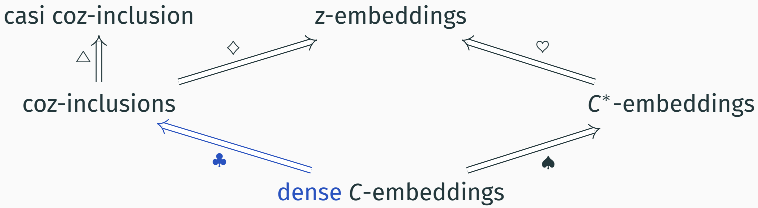

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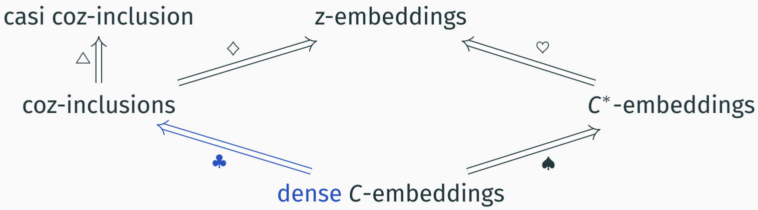

♠: $\mathcal{L}(\mathbb{R}) \subseteq \beta\mathcal{L}(\mathbb{R})$ is C^*- not C-embedded.

◇: $\mathcal{L}(\mathbb{R}) \subseteq \beta\mathcal{L}(\mathbb{R})$ is z-embedded not coz-included

♡: Every proper open sublocale of $\mathcal{L}(\mathbb{R})$ is z-embedded, but not C^*-embedded.

♣: the booleanization of $\mathcal{L}(\mathbb{R})$ is coz-included but not C-embedded (since $\mathcal{L}(\mathbb{R})$ is perfectly normal but not extremally disconnected).

△: The one point compactification of an uncountable discrete space.

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Coz $S \subseteq$ Coz L	**coz-inclusion**
Coz S is a sublattice of L	C^***-embedded** + dense
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bCoz $S \subseteq$ Coz L	**casi coz-inclusion**
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Characterizing Frames with Coz-Inclusions

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Theorem

Let L be a frame, the following are equivalent:

(i) *L is normal.* $(a \vee b = 1 \implies \exists u, v \in L,\ a \vee u = 1 = b \vee v \text{ and } u \wedge v = 0)$

(ii) *Every closed sublocale of L is z-embedded.*

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Theorem

Let L be a frame, the following are equivalent:

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More Characterizations

Let L be a frame.

$\operatorname{Coz} L$ = cozero elements L^{**} := regular elements CL := complemented elements

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Coz L = cozero elements L^{**} := regular elements CL := complemented elements

$a \in L$ is regular iff $a = a^{**}$ a is complemented iff $a \vee a^* = 1$.

More Characterizations

Let L be a frame.

$\operatorname{Coz} L =$ cozero elements $\quad L^{**} :=$ regular elements $\quad \mathcal{C}L :=$ complemented elements

$\mathcal{C}L \subseteq \operatorname{Coz} L \quad$ and $\quad \mathcal{C}L \subseteq L^{**}$

More Characterizations

Let L be a frame.

$\text{Coz } L$ = cozero elements L^{**} := regular elements CL := complemented elements

$CL \subseteq \text{Coz } L$ and $CL \subseteq L^{**}$

ED: extremally disconnected $L^{**} = C\,L$

PN: perfectly normal $\text{Coz } L = L$

boolean: $L = CL$

almB: almost boolean $\text{Coz } L = CL = L^{**}$

P-frame: $\text{Coz } L \subseteq CL$

Almost P: $\text{Coz } L \subseteq L^{**}$

Oz: Oz-frame $L^{**} \subseteq \text{Coz } L$

WPN: weakly perfectly normal Every sublocale is z-embedded

POz: perfectly Oz Every open sublocale is coz-included

F-Frame: Every cozero sublocale is C^*-embedded

Quasi F: Every dense cozero sublocale is C^*-embedded

More Characterizations

Let L be a frame.

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$CL \subseteq \mathrm{Coz}\, L$ and $CL \subseteq L^{**}$

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P-frame: $\mathrm{Coz}\, L \subseteq CL$

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Oz: Oz-frame $L^{**} \subseteq \mathrm{Coz}\, L$

WPN: weakly perfectly normal Every sublocale is z-embedded

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F-Frame: Every cozero sublocale is C^*-embedded

Quasi F: Every dense cozero sublocale is C^*-embedded

Sublocales	dense Cozero	Cozero	Zero	Open	Closed	Dense	All
z-emb.	all frames	all frames	$\star$	Oz	Normal	Oz	WPN
C^*-emb.	quasi F	F-frame	$\star$	ED	Normal	ED	ED+CN
C-emb.	almost P	P-frame	$\star$	almB	Normal	almB	P + WPN
casi coz-incl.	$*$	$\dagger$	all frames	POz	PN	$\circ$	PN
coz-incl.	$*$	$\dagger$	$\star$	POz	PN	$\circ$+Oz	PN

Even More Characterizations

The **boolenization** of a frame L is the smallest dense sublocale of L. It is in fact the sublocale of all regular elements; that is L^{**}.

The following table characterizes frames having their booleanization embedded in a designated way:

Booleanization	**Frame**
z-embedded	coole
casi coz-incl.	casi-Oz
coz-included	Oz
C^*-embedded	ED
C-embedded	almost boolean
isomorphism	boolean

Thank you!