# The *E*-base of finite semidistributive lattices

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May 21, 2025

# Outline

In this talk we plan

- to give the definition of $E$-base as a subset of minimal covers in a lattice;
- discuss its relation to the $D$-base and the canonical base;
- present the main result: the $E$-base in semidistributive lattices;
- discuss the $E$-base in other classes.

# Finite lattices and their join irreducibles

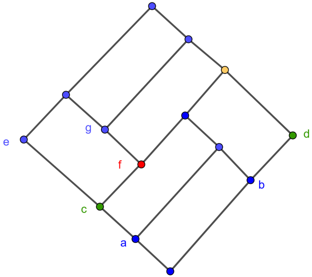



Figure: Finite lattice with seven join-irreducibles

### A lattice

$\langle L, \vee, \wedge \rangle$

### Join irreducible

An element $j$ of lattice such that $j = a \vee b$ implies $j = a$ or $j = b$.

### Example of a finite lattice

Relations on join irreducibles: $f \leq g$, or $f \leq c \vee d$

# Finite lattice representation by ji-relations

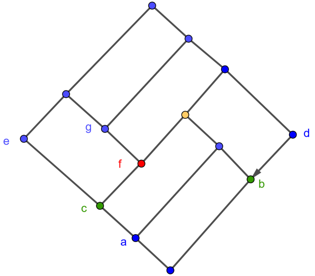



Figure: Relation $f \leq c \vee b$ refines $f \leq c \vee d$

## OD-graph of a finite lattice

(I) Partially ordered set of join irreducibles.
(II) Minimal *join-covers* of join irreducibles.

## Example: part (I)

$a \leq c \leq f \leq g$, $c \leq e$, $b \leq d$

## Example: part (II)

$f \leq c \vee d$ is a join-cover of $f$, but it is not a *minimal* cover, since there is $f \leq c \vee b$, where $b \leq d$ from (I)

# OD-graph as finite lattice representation

J.B.Nation, *An approach to lattice varieties of finite height*
Alg. Universalis (1990)

# Lattice as a closure system

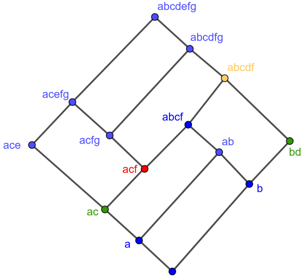



Figure: Closure system on $A = \{a, b, c, d, e, f, g\}$

## Closure

Closure operator: $\phi : 2^A \longrightarrow 2^A$
$\phi(Y) = \{j \in Ji : j \leq \bigvee Y\}$
Closure system:
$\mathcal{F} = \{\phi(Y) : Y \in 2^A\}$

## Closure system perspective

Relation $f \leq g$ becomes:
$f \in \phi(g)$, or the *implication* $g \to f$

## Cover relation

Relation $f \leq c \vee d$ becomes
$f \in \phi(cd)$, or the *implication* $c\&d \to f$

# Implicational bases

Many uses of implicational bases:

- attribute implications in Formal Concept Analysis;
- functional dependencies in database theory;
- (pure) Horn formulas in propositional logic;
- defining relations of join irreducibles in lattice theory.

# The $D$-base of a closure system

KA, J.B.Nation, R. Rand, *Ordered direct implicational basis of a finite closure system*, Disc. Appl. Math. (2013)

**Definition**

A set of implications $\Sigma_D$ of a finite closure system on set $A$ that comprises

- $g \to f$, $f \in \phi(g)$, $g, f \in A$;
- $B \to f$, $f \in \phi(B)$, if $B \subseteq A$ is a *minimal* join cover of $f$.

# Canonical base

J.L.Guigues, V. Duquenne, *Familles minimales d'implications informatives resultant d'un tableau de donnes binares*, Math. et Sci.Humaines (1986)

D. Maier, *Minimal covers in the relational database model*, J.Assoc.Comp. Machinery (1980)

**Theorem**

*Let $(A, \mathcal{F})$ be a finite closure system with closure operator $\phi$.*
*(i) Then a minimum implicational base defining this system is* $\Sigma_{DG} = \{P \to \phi(P) \setminus P : P \text{ is a critical set}\}$;
*(ii) For every other implicational base $\Sigma$ defining the same system, and for every critical set $P$, there exists $S \to T$ in $\Sigma$ such that $S \subseteq P$ and $\phi(S) = \phi(P)$.*

# Canonical base: example

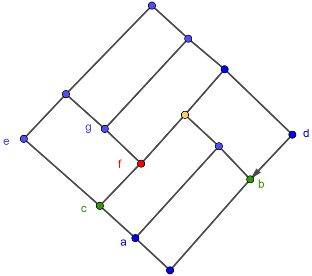



Figure: Implication $abc \to f$ from the Canonical base

**Critical set**

Set $P = \{a, b, c\} \subseteq A$ is critical.
$\phi(abc) = abcf$

**Implication for $P$ in $\Sigma_{DG}$**

$abc \to f$

**Implication from the $D$-base**

$bc \to f$

# Canonical versus the *D*-base in data analysis

KA, J.B.Nation, *Discovery of the D-basis in binary tables based on hypergraph dualization*, Theor. Comp. Sci. (2017)

- The *D*-base can be retrieved from the binary table using the algorithm of sub-exponential complexity.

M. Babin, S. Kuznetsov, *Recognizing pseudo-intents is coNP-complete*, CLA Proceedings (2010)

KA, L. Nourine, S. Vilmin, *Computing the D-base and D-relation in finite closure systems*
arxiv (2024)

- Further improvement of *D*-basis algorithm based on dualization in distributive lattices: K. Elbassioni (2022)

# The $D$-base in lower bounded lattices

KA, J.B.Nation, R. Rand, *Ordered direct implicational basis of a finite closure system*, Disc. Appl. Math. (2013)

**Lemma**

*If $L$ is a finite lower bounded lattice, then a subset of the $D$-base, called the $E$-base forms a valid and minimum base of the corresponding closure system.*

- *Lower bounded lattice*: a lower bounded homomorphic image of a free lattice $\leftrightarrow$ lattice without $D$-cycles.
- Valid part of Lemma follows from Theorem 2.51, *Free lattices*, by R. Freese, J. Jezek, J.B.Nation (1995)

# The $E$-base: example

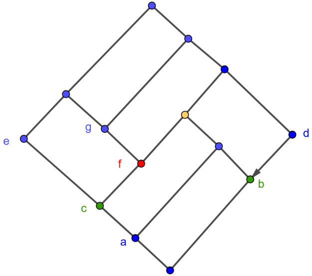



Figure: Implication $bc \rightarrow f$ in the $E$-base

**Implication from the $E$-base**

$bc \rightarrow f$

**Implication from the $D$-base, not in the $E$-base**

$ad \rightarrow f$

**How to decide**

$\phi(bc) \subset \phi(ad)$ and $\phi(bc)$ is minimal among all minimal covers for $f$

# The $E$-base

## Definition

A set of implications $\Sigma_E \subseteq \Sigma_D$ is called the $E$-base:

- $C \to x$ is in the $E$-base iff $\phi(C)$ is a minimal closed set among $\{\phi(B) : B \to x \text{ in } D\text{-base}\}$

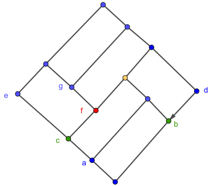



Figure: $bc \to f$ in the $E$-base

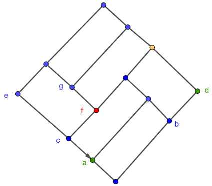



Figure: $ad \to f$ not in the $E$-base

# Essential sets

J.L.Guigues, V. Duquenne, *Familles minimales d'implications informatives resultant d'un tableau de donnes binares*, Math. et Sci.Humaines (1986)

**Definition**

If $P \rightarrow \phi(P)$ is an implications from the canonical basis, then $\phi(P)$ is called *essential*.

# Essential sets: continued

**Definition**

*If $P \to \phi(P) \setminus P$ is an implications from the canonical basis, then $\phi(P)$ is called essential.*

KA, J.B.Nation *On implicational bases of closure systems with unique critical sets*, Disc. Appl. Math. (2017)

- $SD_j$ law: $x \vee y = x \vee z \longrightarrow x \vee y = x \vee (y \wedge z)$

**Theorem**

- *For every finite $SD_j$ lattice there is only one implication $P \to \phi(P)$ in the Canonical basis, for every essential set $\phi(P)$.*
- *If $L$ is a finite lower bounded lattice and $\phi(P)$ is essential, there exists $B \to x$ in the $E$-base such that $\phi(B) = \phi(P)$.*

- Lower bounded lattices are $SD_j$

# $E$-base may not be a valid base

In general, the $E$-base may not have enough implications to represent the closure system.

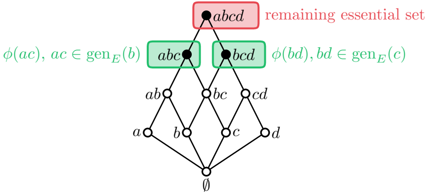



Figure: The leaf

- $ad \to bc$ is in the canonical base;
- no implication in the $E$-base accounts for essential set *abcd*.

# A property of the $E$-base

**Lemma (KA and S.Vilmin, 2025+)**

*For every $B \to x$ in the $E$-base, $\phi(B)$ is essential.*

**Proof.**

Take $P = \phi(B) \setminus \{y \in A : x \in \phi(y)\}$.

- $\phi(P) = \phi(B)$.
- Need to check: If $Q \subset P$ and $\phi(Q) \subset \phi(P)$, then $\phi(Q) \subset P$.
- Fact. *Critical sets are exactly minimal sets with this property in their closure class.*
- Indeed, $x \notin \phi(Q)$, since $\phi(B)$ is the minimal closed set with $x \in \phi(B) \setminus B$.
- If $P$ is not minimal, we could find $P' \subset P$ with this property and such that $\phi(P') = \phi(P) = \phi(B)$. Then $P'$ is critical.

□

# Semidistributive lattices

**Definition**

Lattice is semidistributive, if it satisfies $SD_j$ and the dual law $SD_m$.

Important example: free lattices are semidistributive.

**Theorem (KA, S. Vilmin, 2025+)**

*$E$-basis is valid and minimum in any finite semidstributive lattice.*

**Proof.**

- Let $E = \phi(P)$ be essential set with unique critical set $P$, due to $SD_j$.
- Let $B$ be the canonical join representation of $E$, due to $SD_j$.
- Let $B'$ be the canonical join representation of $E' \prec E$, and let $x \in B' \setminus P$.
- Use $SD_m$ to show that $B \to x$ is in the $E$-base.

# $E$-base divide in the hierarchy of classes

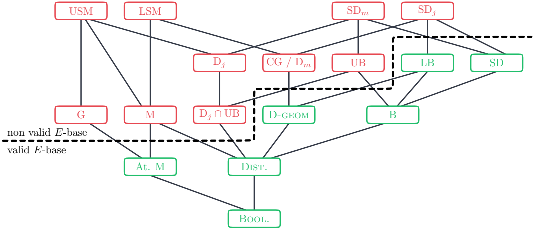



Figure: Some classes identified on the divide

More classes confirmed for the divide:

- Atomistic modular lattice have the valid $E$-base.
- $SD_j$, $SD_m$ (even upper bounded), geometric lattices and modular lattices in general do not have the valid $E$-base.

# Open question: *D*-cycles and the *E*-base

Do implications for elements from *D*-cycles always appear in the *E*-base?

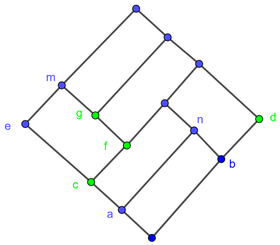



Figure: *D*-cycle in SD lattice

**D-cycle**

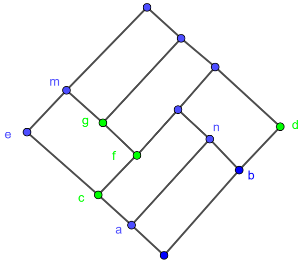
Sequence of join irreducibles:
$g\ D\ f\ D\ c\ D\ d\ D\ g$

**Minimal covers**

$g \leq f \vee e$, $f \leq c \vee b$,
$c \leq d \vee a$, $d \leq g \vee b$

**Implications from the *E*-base**

$fe \to g$, $bc \to f$,
$ad \to c$, $bg \to d$




Thank you!