Algebraic properties of groups in lattice framework

Branimir Šešelja
University of Novi Sad
Joint research with
Andreja Tepavčević and **Jelena Jovanović**

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Concerning groups, weak congruence lattices are extensions of the subgroup lattices: normal subgroups of all subgroups are represented by the corresponding congruences.
It turned out that main properties of groups, related to class operators S, H and P, can be equivalently formulated or represented in lattice terms.
Also the other structural and algebraic properties like series and systems of subgroups, commutator subgroups, center and related notions, have their lattice-theoretic definitions and interpretations.

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Our first results related to weak congruence lattices of groups:

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- G. Czédli, B. Šešelja, A. Tepavčević, *Semidistributive elements in lattices; application to groups and rings*, Algebra Univers. 58 (2008) 349–355.
- G. Czédli, M. Erné, B. Šešelja, A. Tepavčević, *Characteristic triangles of closure operators with applications in general algebra*, Algebra Univers. 62 (2009) 399–418.

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Then we continued:

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- J. Jovanović, B. Šešelja, A. Tepavčević, *Lattice characterization of finite nilpotent groups*, Algebra Univers. 2021 82(3) 1–14.
- J. Jovanović, B. Šešelja, A. Tepavčević, *Lattices with normal elements* Algebra Univers. 2022 83(1) 1–28.
- M.Z. Grulović, J. Jovanović, B. Šešelja, A. Tepavčević, *Lattice Characterization Of Some Classes Of Groups By Series Of Subgroups*, International Journal of Algebra and Computation, 2023 33(02) 211–235.

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- J. Jovanović, B. Šešelja, A. Tepavčević, *Nilpotent groups in lattice framework*, Algebra univers. 2024 85(4) 40.
- M.Z. Grulović, J. Jovanović, B. Šešelja, A. Tepavčević, *On classes of groups characterized by classes of lattices* (submitted).
- M.Z. Grulović, J. Jovanović, B. Šešelja, A. Tepavčević, *Systems of subgroups and Kurosh-Chernikov classes of groups in lattice framework* (submitted).

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Under the set inclusion, $\mathrm{Wcon}(\mathcal{A})$ *is an algebraic lattice in which the diagonal* Δ *(representing the whole algebra* $\mathcal{A}$*) is a codistributive element:*

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If a is a codistributive element in a lattice L, then the mapping $m_a : L \longrightarrow \downarrow a$ defined by $m_a(x) = a \wedge x$ is an endomorphism on L.

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Therefore, under the map $m_\Delta : \mathrm{Wcon}(\mathcal{A}) \longrightarrow \downarrow\Delta$ given by $m_\Delta(x) = \Delta \wedge x$, the lattice $\mathrm{Sub}(\mathcal{A})$ of subalgebras of $\mathcal{A}$ is up to the isomorphism a retract of the lattice $\mathrm{Wcon}(\mathcal{A})$ of weak congruences of this algebra.

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In an algebraic lattice L for a codistributive element a, the classes of the kernel φ_a of m_a have top elements, we denote this by $\overline{x} = \bigvee [x]_{\varphi_a}$, $x \in L$.

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The classes of the kernel φ_Δ of m_Δ are congruence lattices of subalgebras of $\mathcal{A}$:
If $x \in \downarrow \Delta$, then $x = \Delta_B$ for a subalgebra $\mathcal{B}$ of $\mathcal{A}$, and the top element $\overline{[\Delta_B]_{\varphi_\Delta}}$ of the class $[x]_{\varphi_\Delta}$ is the square B^2. Therefore,

$$[\Delta_B]_{\varphi_\Delta} = [\Delta_B, B^2] \cong \mathrm{Con}(\mathcal{B}).$$

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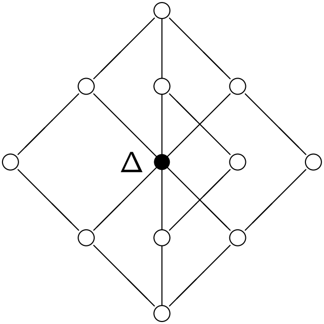
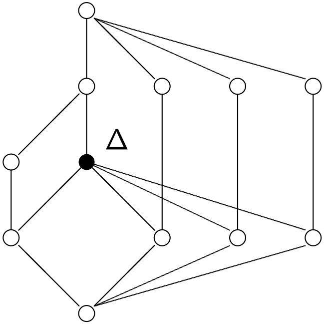


Wcon($\mathcal{K}$)



Wcon($\mathcal{S}_3$)

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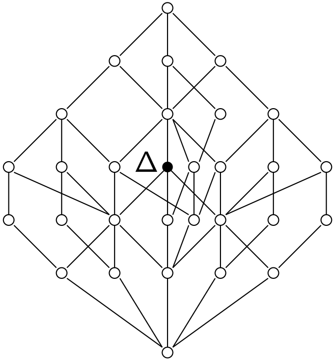
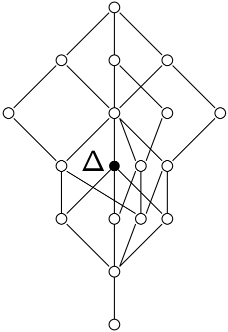


dihedral group of order 8



quaternion group

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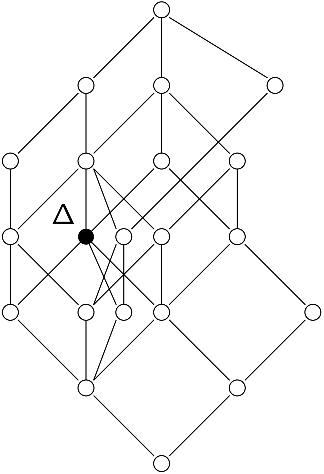

$\mathrm{Wcon}(R)$

Weak congruence lattices of groups

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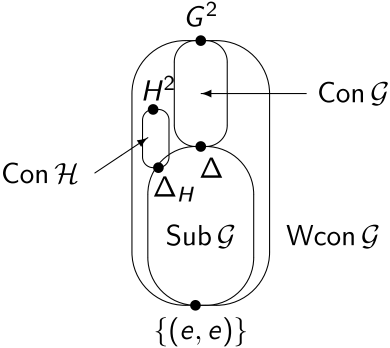

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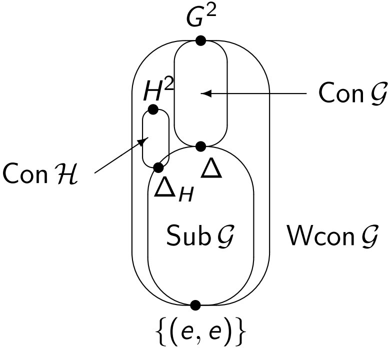
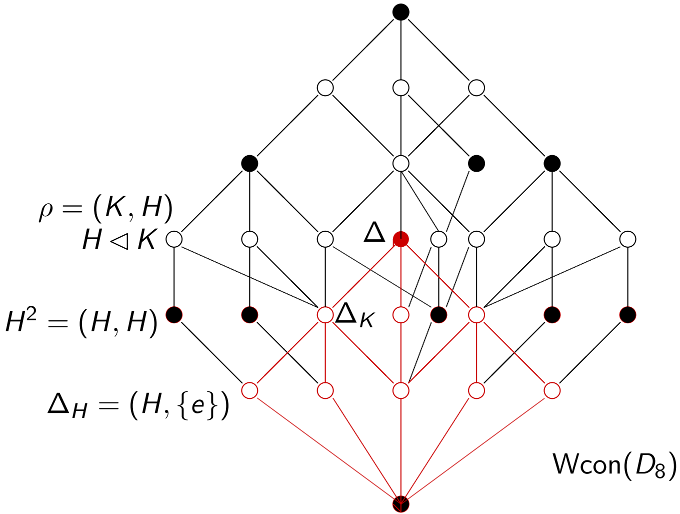


Therefore, for a group G, the lattice $\mathrm{Wcon}(G)$ can be represented as a collection of all ordered pairs (H, K) of subgroups of G, so that $K \lhd H$.



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For every subgroup H of G, represented as the diagonal $\Delta_H \in \downarrow\Delta$,

$$[\Delta_H]_\varphi = [\Delta_H, H^2] \cong \mathrm{Sub_n}(H)$$

under $\theta \to [e]_\theta$. Clearly, $\uparrow\Delta \cong \mathrm{Sub_n}(G)$.

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By $n \blacktriangleleft b$ we denote that n is normal in $\downarrow b$; the sign is filled in, in order to indicate the difference with the normality among groups.

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If G is a group and $\theta \in \mathrm{Wcon}(G)$, then by the previous definition θ_Δ is a diagonal from $\downarrow\Delta$, defined by

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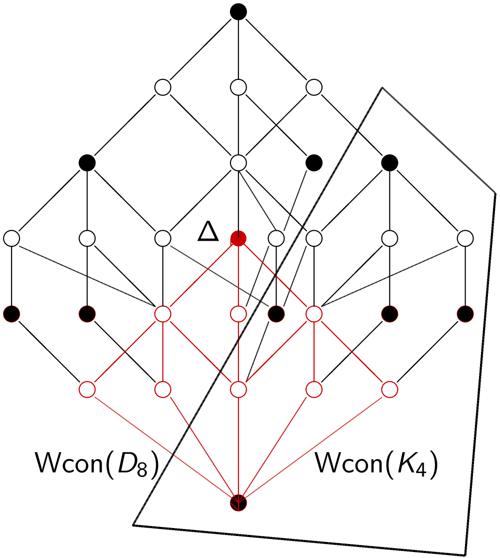
$$H \lhd K \quad \textit{if and only if} \quad [H^2, H^2 \vee \Delta_K] \cap T_\Delta = \{H^2\}$$


Δ
Wcon(D8)
Wcon(K4)

Operator H

Operator H

If H is a subgroup of a group G, then $H \lhd G$, if and only if the principal filter $\uparrow(H^2)$ in $\mathrm{Wcon}(G)$ is the weak congruence lattice of G/H with the main codistributive element $H^2 \vee \Delta$.
Analogously, for subgroups H, K of G with $H \subseteq K$, $H \lhd K$ if and only if the interval $[H^2, K^2]$ in $\mathrm{Wcon}(G)$ is the weak congruence lattice of K/H, with the main codistributive element $H^2 \vee \Delta_K$.

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Therefore, all groups which are homomorphic images of G and of all its subgroups are represented by their weak congruence lattices in $\mathrm{Wcon}(G)$ as the corresponding intervals of squares.

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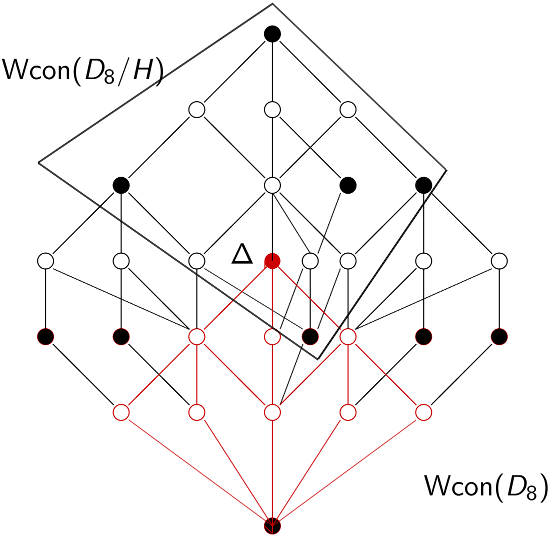
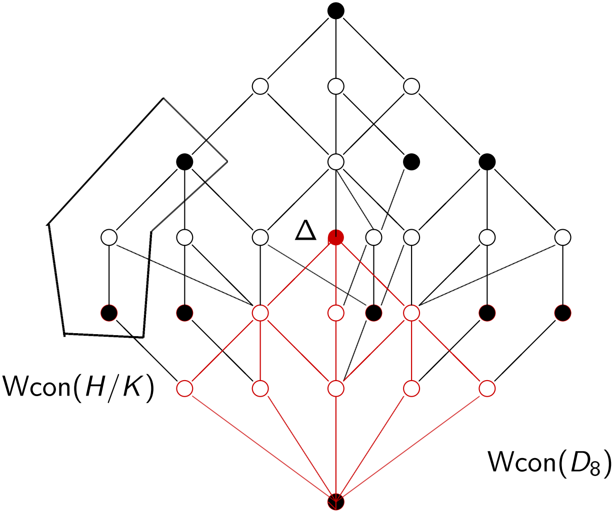

$\mathrm{Wcon}(D_8/H)$
Δ
$\mathrm{Wcon}(D_8)$



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Operator P

Operator P

Theorem

A group G is the internal direct product of subgroups $\{H_i, i \in I\}$ if and only if the following hold in the lattice $\mathrm{Wcon}(G)$:
(i) for every $i \in I$, $\Delta_{H_i} \blacktriangleleft \Delta$;
(ii) $\bigvee_{i \in I} H_i^2 = G^2$; and
(iii) for every $j \in I$, $H_j^2 \wedge \bigvee_{i \neq j} H_i^2 = \{(e,e)\}$.

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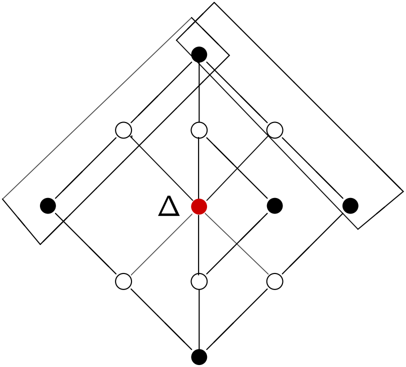

$\mathrm{Wcon}(\mathcal{K})$

L-classes of groups; varieties

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Theorem

An L-class $\mathcal{C}$ of group is a variety if and only if the following hold:
– in the lattice $\mathrm{Wcon}(G)$ *of every* $G \in \mathcal{C}$*, for every* $\Delta_H \in \downarrow\Delta$ *with* $\Delta_H \blacktriangleleft \Delta$*, the filter* $\uparrow(H^2)$ *fulfills the lattice properties* $L_{\mathcal{C}}$*; and*
– every group for which disjoint filters $\uparrow(H^2)$*,* $\Delta_H \blacktriangleleft \Delta$*, fulfill the properties* $L_{\mathcal{C}}$*, also belongs to* $\mathcal{C}$*.*

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Theorem

If an L-class $\mathcal{C}$ of groups is determined by a set of lattice identities which are assumed to be fulfilled by the weak congruence lattices of these groups, then $\mathcal{C}$ is a variety if and only if this class is closed for groups whose filters $\uparrow(H^2)$, $\Delta_H \blacktriangleleft \Delta$ which are disjoint, fulfill these identities.

Abelian and hamiltonian groups

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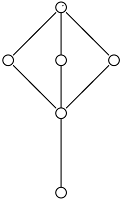

Q

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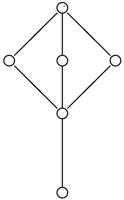

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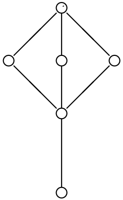

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A group G is Hamiltonian if and only if the lattice $\mathrm{Wcon}(G)$ *is modular, containing an interval sublattice* $[\Delta_H, \Delta_K]$, $H, K \in \mathrm{Sub}(G)$, *isomorphic to the lattice Q.*

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Theorem

A group G is semisimple if and only if in the lattice $\mathrm{Wcon}(G)$ *for every* $\Delta_H \blacktriangleleft \Delta$, $\Delta_H \neq \{(e,e)\}$, *the ideal* $\downarrow H^2$ *is not an A-lattice.*

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The main codistributive element a in L is **perfect**, if $a' = a$.

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A group is **perfect** if it coincides with the commutator subgroup.

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A group is **perfect** if it coincides with the commutator subgroup. G is **metabelian** if G' is abelian.

Theorem

A group G is:

- *Abelian if and only if $\Delta' = \{(e, e)\}$;*
- *perfect if and only if Δ is a perfect element in $\mathrm{Wcon}(G)$, equivalently, if and only if for every subgroup $\Delta_H \in \downarrow\Delta$ such that $\Delta_H \blacktriangleleft \Delta$, $\uparrow H^2$ is not an A-lattice in $\mathrm{Wcon}(G)$.*
- *metabelian if and only if $\downarrow\overline{\Delta'}$ is an A-lattice with the main codistributive element Δ'.*

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Theorem

A group G is hypoabelian if and only if in $\mathrm{Wcon}(G)$ *there is no* $\Delta_H \in {\downarrow}\Delta$, $\Delta_H \blacktriangleleft \Delta$, *for which* ${\uparrow}H^2$ *is an A-lattice.*

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Central series; center; nilpotent groups

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A **central series** of a group G is a finite sequence

$$\{e\} = H_0 \leqslant H_1 \leqslant \ldots, \leqslant H_n = G$$

of normal subgroups of G, such that all factors are *central*, i.e., for every i,

$$H_{i+1}/H_i \leqslant Z(G/H_i).$$

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A group G is **nilpotent** if it has a central series.

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Proposition

A subgroup K of a group G is central if and only if in the lattice $\mathrm{Wcon}(G)$ *for every* $\Delta_X \in \mathrm{C}(\downarrow\Delta)$, *the ideal* $\downarrow\overline{\Delta_K \vee \Delta_X}$ *is an A-lattice determined by* $\Delta_K \vee \Delta_X$.

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Let S be the set of diagonals of central subgroups of G, $S = \{\Delta_K \in \downarrow\Delta \mid \mathrm{K}$ is a central subgroup of $\mathrm{G}\}$. (Obviously, $S \subseteq \downarrow\Delta$ in $\mathrm{Wcon}(G)$ and by the above proposition, it can be defined in lattice-theoretic terms:
$S = \{\Delta_K \in \downarrow\Delta \mid (\forall\Delta_X \in \mathrm{C}(\downarrow\Delta))(\downarrow\overline{\Delta_K \vee \Delta_X}$ is an A-lattice determined by $\Delta_K \vee \Delta_X)\}$.)

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Theorem

A subgroup H of a group G is the center of G ($H = Z(G)$) if and only if $\Delta_H = \bigvee S$ in the lattice $\mathrm{Wcon}(G)$.

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Theorem

A group G is nilpotent if and only if the lattice $\mathrm{Wcon}(G)$ has a finite series of intervals

$$[\{e\}^2, H_1^2], [H_1^2, H_2^2], \ldots, [H_i^2, H_{i+1}^2], \ldots, [H_k^2, G^2],$$

so that for every $i \in \{0, 1, \ldots, k\}$ the following holds:

(a) $\Delta_{H_i} \blacktriangleleft \Delta$;

(b) *in the sublattice $[H_i^2, G^2]$ as a lattice with normal elements determined by $H_i^2 \vee \Delta$, for every $\delta \in \mathrm{C}([H_i^2, H_i^2 \vee \Delta])$, the interval $[H_i^2, \overline{H_i^2 \vee \Delta_{H_{i+1}} \vee \delta}]$ is an A-lattice determined by $H_i^2 \vee \Delta_{H_{i+1}} \vee \delta$.*

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The **normal closure** of a subgroup H of G, denoted b H^G, is the smallest normal subgroup of G containing H.
If a group has a maximal subgroup that is also core-free, then it is termed a **primitive** group.

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Proposition

If H is a subgroup of a group G, then $H = \mathrm{soc}(G)$ if and only if
$\Delta_H \blacktriangleleft \Delta$ *and*
$H^2 \vee \Delta = \bigvee_i \{\theta_i \mid \theta_i$ *are atoms in the filter* $\uparrow\!\Delta$ *of* $\mathrm{Wcon}(G)\}$.

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Proposition

Let H be a subgroup of a group G. Then in the lattice $\mathrm{Wcon}(G)$:

$$\Delta_{H_G} = \bigvee_i \{\Delta_{N_i} \mid \Delta_{N_i} \blacktriangleleft \Delta \text{ and } \Delta_{N_i} \leqslant \Delta_H\} \tag{1}$$

$$\Delta_{H^G} = \bigwedge_j \{\Delta_{N_j} \mid \Delta_{N_j} \blacktriangleleft \Delta \text{ and } \Delta_H \leqslant \Delta_{N_j}\} \tag{2}$$

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Proposition

Let G be a group which possesses minimal normal subgroups. Then a subgroup H of G is the socle if and only if $H^2 \vee \Delta = \bigvee_i \theta_i$, where $\theta_i, i \in I$ are atoms in the filter-sublattice $\uparrow\Delta$ of the lattice $\mathrm{Wcon}(G)$.

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Theorem

A group G is a primitive group if and only if there is a square M^2 in the lattice $\mathrm{Wcon}(G)$ such that for every atom θ in the filter-sublattice $\uparrow\Delta$ of the lattice $\mathrm{Wcon}(G)$, we have $M^2 \vee \theta = G^2$.

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An Abelian group G is said to be **elementary** *if* $\mathrm{soc}(G) = G$. *Therefore, G is elementary if and only if in the lattice* $\mathrm{Wcon}(G)$,

$$G^2 = \bigvee_i (\theta_i \mid \Delta \prec \theta_i).$$

Series and systems of subgroups

Series and systems of subgroups

A finite series of subgroups
$\{e\} = H_0 < H_1 < \ldots < H_{n-1} < H_n = G$, such that for every $1 \leqslant i \leqslant n$, $H_{i-1} \lhd H_i$ is a **subnormal series of G**.

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If subgroups in a subnormal series are normal in G, then the above series is a **normal series of G**.

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Subnormal series corresponds to a chain
$\{(e, e)\} < \Delta_{H_1} < \ldots < \Delta_{H_{n-1}} < \Delta_{H_n} = \Delta$, *such that for every* $1 \leqslant i \leqslant n$, $\Delta_{H_{i-1}} \blacktriangleleft \Delta_{H_i}$.

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Subnormal series corresponds to a chain
$\{(e,e)\} < \Delta_{H_1} < \ldots < \Delta_{H_{n-1}} < \Delta_{H_n} = \Delta$, *such that for every* $1 \leqslant i \leqslant n$, $\Delta_{H_{i-1}} \blacktriangleleft \Delta_{H_i}$.
Normal series corresponds to the above chain with the additional property that for every i, $\Delta_{H_i} \blacktriangleleft \Delta$.

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Subnormal and normal chains are often related to particular chains of intervals, e.g., as for solvable groups:

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Theorem

A group is **solvable** *if if and only if in* $\mathrm{Wcon}(G)$ *there is a subnormal chain* $\{(e,e)\} < \Delta_{H_1} < \ldots < \Delta_{H_{n-1}} < \Delta_{H_n} = \Delta$, *such that the chain of intervals*

$$[\{(e.e)\}, H_1^2], [H_1^2, H_2^2], \ldots, [H_{n-1}^2, G^2]$$

consists of A-lattices.

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A group G is said to be an M-**group** if it contains a subnormal series whose factors are eider infinite cyclic or finite groups.

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A group G is an M-group if and only if the lattice $\mathrm{Wcon}(G)$ contains a subnormal series $\{e\} = H_0 < H_1 < \ldots < H_{n-1} < H_n = G$, such that the corresponding series of intervals $[\{(e.e)\}, H_1^2], [H_1^2, H_2^2], \ldots, [H_{n-1}^2, G^2]$ consists of finite or Z-sublattices.

A group G is said to be an ***M*-group** if it contains a subnormal series whose factors are eider infinite cyclic or finite groups.

A group G is an M-group if and only if the lattice $\mathrm{Wcon}(G)$ *contains a subnormal series* $\{e\} = H_0 < H_1 < \ldots < H_{n-1} < H_n = G$, *such that the corresponding series of intervals* $[\{(e.e)\}, H_1^2], [H_1^2, H_2^2], \ldots, [H_{n-1}^2, G^2]$ *consists of finite or Z-sublattices.*

(Z-lattice is the weak congruence lattice of the group $(Z, +)$: $(\{(m, n) \mid m, n \in \mathbb{N}_0, n \text{ divides } m\}, \sqsupseteq)$, where $\mathbb{N}_0$ are nonnegative integers and the order $\sqsupseteq$ is componentwise, dual to divisibility and for every $n \in \mathbb{N}_0$, $0 \sqsupseteq n$.)

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Examples of Kurosh-Cernikov classes of groups

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SN-groups; each of them containing a solvable subnormal subgroup system.

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SN-groups; each of them containing a solvable subnormal subgroup system.
SN^*-groups; they have a well-ordered ascending solvable subnormal subgroup system.

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Theorem

A group G is an SN-group if and only if in the lattice $\mathrm{Wcon}(G)$ *there is a chain $\mathcal{C}_w$ as a complete sublattice of* $\downarrow\Delta$, *such that the following hold:*
(i) $\{(e,e)\}, \Delta \in \mathcal{C}_w$;
(ii) *for every $\Delta_H \in \mathcal{C}_w$ such that $\Delta_H < \Delta_{H^u}$, the interval* $[H^2, (H^u)^2]$ *is an A-lattice determined by* $H^2 \vee \Delta_{H^u}$.
G is an SN*-group if and only if in the lattice $\mathrm{Wcon}(G)$ *there is a chain $\mathcal{C}_w$ satisfying the conditions above with an additional condition, $\Delta_H < \Delta_{H^u}$, for every $\Delta_H \in \mathcal{C}_w$ such that $\Delta_H < \Delta$.*

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Thanks for watching!