Reductions on Equivalence Relations Generated by Universal Sets

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Outline

Reduction

Definition

Let E,F be two equivalence relations on X, Y respectively, a function $f : X \to Y$ is a **reduction** from E to F if

$$x_1 E x_2 \iff f(x_1) F f(x_2)$$

for all $x_1, x_2 \in X$.

Fact (AC)

The choice function $f : X/E \to X$ is a reduction from $id(X/E)$ to E, and $E \leq F$ if and only if X/E embeds into Y/F.

Things become interesting when imposing the definability of the reduction function.

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Polish space

Definition

Polish space:a separable, completely metrizable topological space.

From now on, X, Y are Polish spaces.

Example

(1) separable Banach spaces: $\mathbb{R}^n{}_{std}$, $l^p(1 \leq p < \infty)$, $(C[0,1], \|\cdot\|_{sup})$.

(2) $[0,1]_{std}$, $(K([0,1]), d_H)$.

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Borel hierarchy

Definition

$\mathbf{B}(X)$: *Borel sets* of X is the σ-algebra generated by the open sets of X.

$$\mathbf{\Sigma}^0_1 = \text{open}, \quad \mathbf{\Pi}^0_1 = \text{closed};$$

for $1 \leq \alpha < \omega_1$,

$$\mathbf{\Sigma}^0_\alpha = \{\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_{\alpha_n}, \alpha_n < \alpha\};$$

$$\mathbf{\Pi}^0_\alpha = \text{ the complements of } \mathbf{\Sigma}^0_\alpha \text{ sets;}$$

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Below $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$

Let $A \subseteq X$,

Definition

$\mathbf{\Sigma}^1_1$ **set**: A is $\mathbf{\Sigma}^1_1$ if it is a continuous image of some Polish space.
$\mathbf{\Pi}^1_1$ **set**: A is $\mathbf{\Pi}^1_1$ if $X \setminus A$ is $\mathbf{\Sigma}^1_1$.
$\sigma(\mathbf{\Sigma}^1_1)$ **set**: the σ-algebra generated by the $\mathbf{\Sigma}^1_1$ set.
$\mathbf{\Sigma}^1_2$ **set**: A is $\mathbf{\Sigma}^1_2$ if it is a continuous image of some $\mathbf{\Sigma}^1_1$ set.
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$\mathbf{\Delta}^1_2$ **set**: A is $\mathbf{\Delta}^1_2$ if it is both $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$.

Theorem (Suslin)

$A \subseteq X$ is Borel iff it is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

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BP sets

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nowhere dense: if the closure of A has no interior,
meager: if A is a countable union of nowhere dense sets.

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various reduction

Definition

Assume X, Y are Polish spaces, Γ is a pointclass on X, a function $f : X \to Y$ is called Γ-measurable if $f^{-1}(U) \in \Gamma$ for each open set $U \in \Gamma$.

Let E, F be equivalence relations on X, Y respectively.

$E \leq_B F$: there is a Borel reduction from E to F;

$E \leq_{\sigma(\Sigma^1_1)} F$: there is a $\sigma(\Sigma^1_1)$-measurable reduction from E to F;

$E \leq_{\sigma(\Delta^1_1)} F$: there is a $\sigma(\Delta^1_1)$-measurable reduction from E to F;

$E \leq_{BP} F$: there is a Baire reduction from E to F.

Also, we denote $E <_\Gamma F$ if $E \leq_\Gamma F$ and $\neg(F \leq_\Gamma E)$, denote $E \sim_\Gamma F$ if $E \leq_\Gamma F$ and $F \leq_\Gamma E$, for pointclass Γ.

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classical equivalence relations

Definition

E is a **smooth equivalence relation** if $E \leq_B id(\mathbb{R})$.

E_0 on the Baire space ω^ω is defined by:

$$xE_0y \iff \exists n \forall m \geq n(x(m) = y(m))$$

Theorem (Harrington-Kechris-Louveau)

If E is a Borel equivalence relation on X, then exactly one of the following is true:

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Relationships between various reductions & examples

It's trivial that:

$$E \leq_B F \Longrightarrow E \leq_{\sigma(\mathbf{\Sigma}^1_1)} \Longrightarrow E \leq_{BP} F$$

But consider:

$$E \leq_B F \Longleftrightarrow E \leq_{\sigma(\mathbf{\Sigma}^1_1)} F?$$

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1st Example

Theorem (Harrington-Kechris-Louveau)

Let E be a Borel equivalence relation, then

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Relationships between various reductions & examples

2nd example

Recall that $E_\infty \subseteq 2^{\mathbb{F}_2} \times 2^{\mathbb{F}_2}$ is the equivalence relation defined by:

$$xE_\infty y \iff \exists z \in \mathbb{F}_2(x = zy)$$

Fact

E_∞ is a universal countable equivalence relation and $E_0 <_B E_\infty$.

Theorem (Sullivan-Weiss-Wright)

$$E_0 \sim_{BP} E_\infty$$

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3rd example

Given an equivalence relation E on X, the Friedman-Stanley jump E^+ on X^ω is defined by:

$$xE^+y \iff \forall n \exists m[x(n)Ey(m)] \wedge \forall i \exists j[x(j)Ey(i)].$$

Theorem (Friedman-Stanley)

If E is Borel and has more than one equivalence classes, then $E <_B E^+$.

$$E <_{\sigma(\mathbf{\Sigma}^1_1)} E^+? \quad E <_{BP} E^+?$$

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Outline

Universal Set

Definition

Let Γ be a pointclass of Y, and $A \subseteq X \times Y$. We say A is *universal* for Γ if $\Gamma = \{A_x : x \in X\}$.

If also $x \neq y$ implies $A_x \neq A_y$, we say A is uniquely universal for Γ.

Theorem (folklore)

Every Borel pointclass $\mathbf{\Sigma}^0_\xi$ ($\mathbf{\Pi}^0_\xi$) has a Borel universal.

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Complexity of E_A

For $A \subseteq X \times Y$, denote $A_x = \{y : (x, y) \in A\}$.

Definition

For any set $A \subseteq X \times Y$, define an equivalence relation E_A on X as

$$xE_Ax' \iff A_x = A_{x'}.$$

Theorem

Let $A \subseteq X \times Y$ be a $\mathbf{\Sigma}^1_n$ set universal for all nonempty closed subsets of Y. Then E_A is $\mathbf{\Pi}^1_{n+1}$ and $E_A \leq_{\sigma(\mathbf{\Sigma}^1_n)} \mathrm{id}(2^\omega)$.
If A is Borel, then E_A is $\mathbf{\Pi}^1_1$.

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Let $A \subseteq X \times Y$ be a Σ^1_n set universal for all nonempty closed subsets of Y. Then E_A is $\mathbf{\Pi}^1_{n+1}$ and $E_A \leq_{\sigma(\mathbf{\Sigma}^1_n)} \mathrm{id}(2^\omega)$.
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Fact

There exists a Borel set universal for all the countable subsets of Polish space X

Let $A \subseteq X \times Y$ be universal for countable subsets of Y.

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(1) *A is $\mathbf{\Sigma}^1_1$, then E_A is $\sigma(\mathbf{\Sigma}^1_1)$, $E_A \leq_{\sigma(\mathbf{\Sigma}^1_1)} =^+$ and $=^+ \leq_{\mathbf{\Delta}^1_2} E_A$.*

(2) *A is Borel, then E_A is Borel, $E_A \leq_B =^+$ and $=^+ \leq_{\sigma(\mathbf{\Sigma}^1_1)} E_A$.*

$$=^+ \leq_B E_A? \qquad =^+ \leq_{BP} E_A?$$

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Independent Results on Δ^1_2-reduction of E_A

Fact

Assume $V = L$. If $E \subseteq X \times X$ is a $\mathbf{\Delta}^1_2$ equivalence relation, then $E \leq_{\mathbf{\Delta}^1_2} id(\mathbb{R})$

Let $A \subseteq X \times Y$ be a Σ^1_1 set universal for countable sets of Y.

Corollary

Assume $V = L$. Then $E_A \leq_{\mathbf{\Delta}^1_2} id(\mathbb{R})$.

Theorem

Assume that every $\mathbf{\Delta}^1_2$ set has BP. If E is a $\mathbf{\Sigma}^0_3$ equivalence relation(say, $id(\mathbb{R})$, E_0, E_1, E_∞),then $E_A \not\leq_{\mathbf{\Delta}^1_2} E$.

End!

Independent Results on $\mathbf{\Delta}^1_2$-reduction of E_A

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