Correspondence, Canonicity, and Model Theory for Monotonic Modal Logics

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BLAST 2019

History

Theorem (Fine)

Suppose a variety V of Boolean algebras with operators (BAOs) is elementarily generated, i.e., generated by the duals of members of a class of Kripke frames definable in first-order logic.
Then V is canonical, i.e., closed under canonical extensions.

Our goal is to extend this for monotonic Boolean expansions (BAMs).

Definition

A BAM $(B, \Box)$ is a Boolean algebra B expanded with an operation $\Box : B \to B$ that is monotonic, i.e., for $x, y \in B$, $\Box x \leq \Box y$ whenever $x \leq y$.

Complex Duality for BAMs

Definition

A monotonic neighborhood frame is a pair (F, N^F) of a set F and a neighborhood function $N^F : F \to \mathscr{P}(\mathscr{P}(F))$ s.t. for every $w \in F$ the family $N^F(w)$ is closed under supersets. A member of $N^F(w)$ is a neighborhood of w.

Definition

The underlying BAM [a] F^+ of a monotonic neighborhood frame F is the BAM $(\mathscr{P}(F), \Box^F)$, where

$$\Box^F(X) = \{w \in F \mid X \in N^F(w)\}$$

[a] also known as the dual or the complex algebra

A First-Order-Like Language for Nbhd Frames

Definition (Chang; Litak et al.)

The (empty) language $L_=$ of coalgebraic predicate logic (CPL) has equations and is closed under Boolean combinations, existential quantification, and formation of formulas of the form

$$x \Box_y \phi$$

where $\phi \in L_=$, x is a term, and y is a variable.
We define

$$F \models w \Box_y \phi(y) \iff \phi(F) \in N^F(w)$$

where

$$\phi(F) = \{v \in F \mid F \models \phi(v)\}.$$

CPL is compact and admits the upward and downward Löwenheim-Skolem Theorems and the omitting type theorem.

A Topological Example

For a topological space $X = (X, \tau)$, we associate a monotonic neighborhood frame $X^* = (X, N)$ defined by

$$U \in N(w) \iff w \in U^\circ.$$

(X^* is a topological neighborhood frame.)
The specialization preorder of X is the preorder $\lesssim$ on X defined by $x \lesssim y \iff x \in \overline{\{y\}}$. It is "definable" in $L_=$:

$$x \lesssim y \iff X^* \models \neg(x \,\Box_z\, z \neq y).$$

Hence, there is an $L_=$-sentence ϕ s.t. for topological spaces X

$$X^* \models \phi \iff X \text{ is } \mathrm{T}_0$$

i.e., the *-image of the class of T_0 spaces is CPL-elementary relative to the class of topological neighborhood frames. The same goes for T_1 spaces.

Modal-Logical Examples

For an equation α in the language of BAMs of the form

$$\langle \text{purely Boolean positive term} \rangle \rightarrow \langle \text{positive term} \rangle = 1 \tag{1}$$

there exists an $L_=$-sentence ϕ (a correspondent of α) s.t. for monotonic neighborhood frames

$$F^+ \models \alpha \iff F \models \phi.$$

(One uses the "minimum valuation argument.")
It follows from Hansen's result that such an α defines a complete variety of BAMs.

Canonicity

The notion of canonical extensions for BAOs has been generalized for BAMs in a couple of different ways.

Definition

Let $\mathbb{A} = (A, \Box)$ be a BAM. The (lower) canonical extension $\mathbb{A}^\sigma = (A^\sigma, \Box^\sigma)$ of $\mathbb{A}$ is the canonical extension of the Boolean algebra A expanded by the function $\Box^\sigma$, where

$$\Box^\sigma(u) = \bigvee_{u \supseteq x \in K(A^\sigma)} \bigwedge_{x \subseteq a \in A} \Box(a).$$

Definition

A variety of BAMs is canonical if it is closed under canonical extensions.

Result

Theorem

Let $\mathcal{K}$ be a class CPL-elementary relative to the class of monotonic neighborhood frames. The variety of BAMs generated by $\{F^+ \mid F \in \mathcal{K}\}$ is canonical.

There are several other classes relative to which $\mathcal{K}$ can be CPL-elementary for the result to still obtain (e.g., the class of topological neighborhood frames).

A Consequence

Reconsider an arbitrary equation α of the form

$$\langle \text{purely Boolean positive term} \rangle \to \langle \text{positive term} \rangle = 1 \tag{1}$$

and a correspondent ϕ of α:

$$\mathcal{K} := \{F \mid F^+ \models \alpha\} = \{F \mid F \models \phi\}.$$

Recall that the variety V of BAMs defined by α is generated by $\{F^+ \mid F \in \mathcal{K}\}$ (completeness of V).
By the Theorem, such a variety is canonical.

Proof of the Theorem I

We use the following lemma,
an analogue of what van Benthem used to Fine's theorem:

Lemma

For a monotonic neighborhood frame F, there exists another G s.t.

- *F and G satisfies the same $L_=$ sentences and*
- *there is an embedding $F^{+\sigma} \hookrightarrow G^{+}$.*

One can impose more closure conditions on F and G;
e.g., if F is a topological neighborhood frame, so is G.

Proof of the Theorem II

We basically follow van Benthem's model-theoretic proof.

Proof.

1. Expand F by a predicate for each subset of F.
2. Obtain G by "$\aleph_0$-saturating" F.
3. G may not even be a monotonic nbhd frame. So tweak G:
   1. Remove indefinable neighborhoods from G.
   2. Close off each $N^G(w)$ by intersections of some sort.
   3. Close off each $N^G(w)$ upward.
4. F and G will still satisfy the same $L_=$-sentences.
5. Consider the (surjective) function
   that assigns to each $w \in G$ the "type" realized by w.
6. The function induces an embedding $F^{+\sigma} \hookrightarrow G^+$
   between the dual objects.

□