On the Descending Central Series of Higher Commutators in Simple Algebras

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Preliminaries

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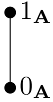
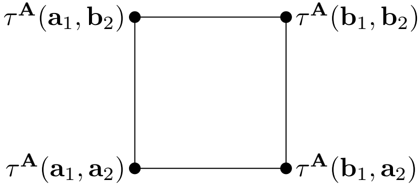
- An algebra, $\mathbf{A}$, is *simple* if its only congruences are $0_{\mathbf{A}}$ and $1_{\mathbf{A}}$.
- The congruences of an algebra $\mathbf{A}$ form a lattice ordered by $\subseteq$.
  - The congruence lattice for a simple algebra $\mathbf{A}$ is



Binary Commutator $[\alpha, \beta]$

For congruences α, β we let $M(\alpha, \beta)$ be the set of all squares of the form:



such that

- τ is a term in the language of $\mathbf{A}$
- $\mathbf{a}_1 \equiv_\alpha \mathbf{b}_1$
- $\mathbf{a}_2 \equiv_\beta \mathbf{b}_2$

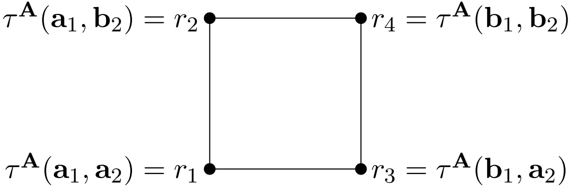
A satisfies the α, β*–term condition* modulo δ if for any square as above in $M(\alpha, \beta)$, we have that

- $\tau^{\mathbf{A}}(\mathbf{a}_1, \mathbf{a}_2) \equiv_\delta \tau^{\mathbf{A}}(\mathbf{a}_1, \mathbf{b}_2)$ implies $\tau^{\mathbf{A}}(\mathbf{b}_1, \mathbf{a}_2) \equiv_\delta \tau^{\mathbf{A}}(\mathbf{b}_1, \mathbf{b}_2)$

The *binary commutator* of α and β, $[\alpha, \beta]$, is the smallest δ for which this holds.

$[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$

We call an element of $M(1_{\mathbf{A}}, 1_{\mathbf{A}})$ a *term square*. Thus a term square is a square of the form:



where $\tau(\mathbf{x}, \mathbf{y})$ is a term in the language of $\mathbf{A}$ and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1$, and $\mathbf{b}_2$ are tuples of values from A. We write

$$S_{\tau^{\mathbf{A}}}(\mathbf{a}_i, \mathbf{b}_i) = (r_1, r_2, r_3, r_4)$$

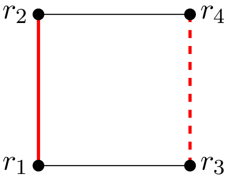
to represent the above term square.

$[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$

An algebra fails the *2–dimensional term condition* if there are tuples $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$, and a term $\tau(\mathbf{x}, \mathbf{y})$ such that the term square

$$S_{\tau^{\mathbf{A}}}(\mathbf{a}_i, \mathbf{b}_i) = (r_1, r_2, r_3, r_4)$$

has equality on the bold red edge and inequality on the dashed red edge in the display below.



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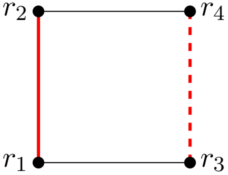
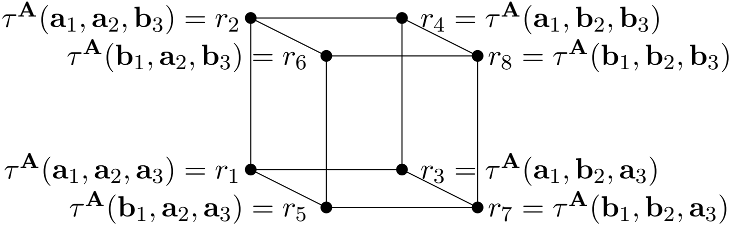


$[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$ if and only if $\mathbf{A}$ satisfies the 2–dimensional term condition.

$[1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$

A *term cube* is a cube of the form:



where $\tau(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a term in the language of $\mathbf{A}$. We write

$$C_{\tau^{\mathbf{A}}}(\mathbf{a}_i, \mathbf{b}_i) = (r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8)$$

to represent the above term cube.

$[1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$

An algebra fails the *3–dimensional term condition* if there are tuples $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, and a term $\tau(\mathbf{x}, \mathbf{y}, \mathbf{z})$ such that the term cube

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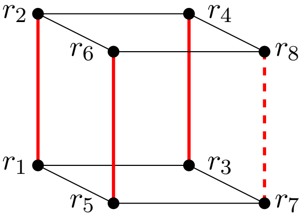

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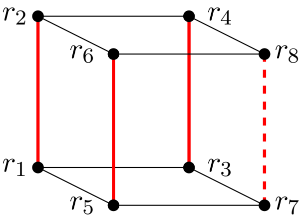

$[1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$ if and only if $\mathbf{A}$ satisfies the 3–dimensional term condition.

Some Facts

- For a simple algebra $\mathbf{A}$:
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- $[\alpha_1, \ldots, \alpha_n, \alpha_{n+1}] \leq [\alpha_1, \ldots, \alpha_n]$
- $[\underbrace{1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}, 1_{\mathbf{A}}}_{(n+1) \text{ many}}] \leq [\underbrace{1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}}_{n \text{ many}}]$

The Descending Central Series of Higher Commutators

$$1_{\mathbf{A}}$$

$$[1_{\mathbf{A}}, 1_{\mathbf{A}}]$$

$$[1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}]$$

$$[1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}]$$

$$\vdots$$

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$$\vdots$$

A weakly descending chain in the congruence lattice of **A**.

Guiding Question

- Can any weakly descending chain be the descending central series of higher commutators?

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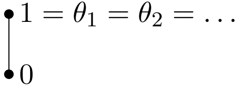
- Given an arbitrary algebraic lattice, L, and a weakly descending chain

$$1 = \theta_1 \geq \theta_2 \geq \theta_3 \geq \dots$$

can this chain be the descending central series of higher commutators for some algebra $\mathbf{A}$:

$$\theta_n = [\underbrace{1_{\mathbf{A}}, \dots, 1_{\mathbf{A}}}_{n \text{ many}}]$$

The Simple Case



The Simple Case

$1 = \theta_1 = \theta_2 = \dots$

0

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$1 = \theta_1 = \theta_2 = \cdots = \theta_n$

$0 = \theta_{n+1} = \theta_{n+2} = \dots$

Higher Commutators in the Lattice of a Simple Algebra

$1_{\mathbf{A}} = [1_{\mathbf{A}}, 1_{\mathbf{A}}] = [1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}] = \dots$

$0_{\mathbf{A}}$

Higher Commutators in the Lattice of a Simple Algebra

$1_{\mathbf{A}} = [1_{\mathbf{A}}, 1_{\mathbf{A}}] = [1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}] = \dots$

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A_5

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A_5

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$\mathbb{Z}_2$

Higher Commutators in the Lattice of a Simple Algebra

$1_{\mathbf{A}} = [1_{\mathbf{A}}, 1_{\mathbf{A}}] = [1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}] = \dots$

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A_5

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$\mathbb{Z}_2$

$$1_{\mathbf{A}} = [1_{\mathbf{A}}, 1_{\mathbf{A}}] = \cdots = [\underbrace{1_{\mathbf{A}}, \dots, 1_{\mathbf{A}}}_{n \text{ many}}]$$

$$0_{\mathbf{A}} = [\underbrace{1_{\mathbf{A}}, \dots, 1_{\mathbf{A}}, 1_{\mathbf{A}}}_{(n+1) \text{ many}}] = \dots$$

The Result

Theorem (W.)

For any natural number $n \geq 2$ there is a simple algebra $\mathbf{A}$ such that

- $[\underbrace{1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}}_{n \text{ many}}] = 1_{\mathbf{A}}$
- $[\underbrace{1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}, 1_{\mathbf{A}}}_{(n+1) \text{ many}}] = 0_{\mathbf{A}}$

Want **A** such that

- **A** is simple
- $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 1_{\mathbf{A}}$
- $[1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$

Define $\mathbf{A}_0$:

$A_0 = B \cup \{c, d_1, d_2, d_3\}$

The Construction for n = 2

Defining $\mathbf{A}_0$

Define $\mathbf{A}_0$:

$A_0 = B \cup \{c, d_1, d_2, d_3\}$

Base:

$$B = \left\{ \begin{array}{lllll} a_1 = a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \dots \\ a_2 = a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & \dots \\ b_1 = b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \dots \\ b_2 = b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \dots \end{array} \right\}$$

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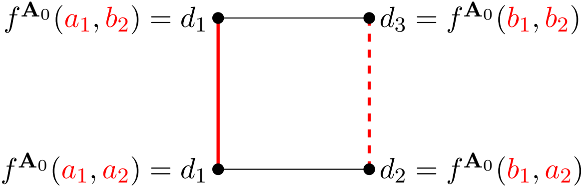
Core:

$C = \{a_1, a_2, b_1, b_2\}$

Define a binary partial operation $f^{\mathbf{A}_0}$:



$f^{\mathbf{A}_0}$ is defined to ensure we have a failure of the 2-dimensional term condition.

Define $\mathbf{A}_{i+1}$:

$$A_{i+1} = A_i \cup \{(x, y, i) \mid (x, y) \in A_i^2 \setminus \mathrm{Dom}(f^{\mathbf{A}_i})\}$$

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$$f^{\mathbf{A}_{i+1}}(x, y) = \begin{cases} f^{\mathbf{A}_i}(x, y) & \text{if } (x, y) \in \mathrm{Dom}(f^{\mathbf{A}_i}) \\ (x, y, i) & \text{if } (x, y) \in A_i^2 \setminus \mathrm{Dom}(f^{\mathbf{A}_i}) \end{cases}$$

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$$\mathrm{Dom}(f^{\mathbf{A}_{i+1}}) = A_i^2$$

The Construction for n = 2

Defining **A**

Define **A** :

$A = \bigcup_{i \in \omega} A_i$

$f^{\mathbf{A}} = \bigcup_{i \in \omega} f^{\mathbf{A}_i}$

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For each $(p, q, r) \in (A \setminus B)^3$ with p, q, r pairwise distinct, add a unary operation:

$$u^{\mathbf{A}}_{p,q,r} = (p\ q\ r)(a_{1,0}\ a_{1,1} \dots)(a_{2,0}\ a_{2,1}\ \dots)(b_{1,0}\ b_{1,1}\ \dots)(b_{2,0}\ b_{2,1}\ \dots)$$

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Finally add the unary operation:

$$u^{\mathbf{A}} = (a_1\ a_2\ b_1\ b_2\ c)$$

Thank You