Expressivity in some many-valued modal logics.

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Table of contents

Introduction

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(partially) why? offer a much higher expressive power than CPL and (generally) much lower complexity than FOL (most well-known and used modal logics are decidable).

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(again) Richer logics, but many well-known infinitely-valued cases still decidable (Ł, Gödel, Product, H-BL...).

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- Intuitionistic modal logics are particularly "nice": they naturally enjoy a relational semantics with an intuitive meaning.
- what about the rest? a seemingly reasonable approach: valuation of Kripke models/frames over classes of algebras
  - Some modal MV logics have been axiomatised, but most have not. [Many usual intuitions fail, and usual constructions need to be adapted to get completeness.]
  - Relation to purely relational semantics is unknown.
  - Tools from classical modal logic like Sahlqvist theory have not been developed (wider set of operations + more specific semantics...)
  - ...

Some definitions

The non-modal part

Definition

A (integral commutative bounded) **Residuated Lattice A** is $\langle A, \odot, \to, \wedge, \vee, 0, 1\rangle$ such that

- $\langle A, \wedge, \vee\rangle$ is a lattice,
- $\langle A, \odot, 1\rangle$ is a commutative monoid
- $x \odot y \leq z \iff x \leq y \to z$ (residuation law)
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$\Gamma \models_{\mathcal{C}} \varphi$ iff for any $\mathbf{A} \in \mathcal{C}$ and any $h \in Hom(Fm, \mathbf{A})$, if $h(\Gamma) \subseteq \{1\}$ then $h(\varphi) = 1$.

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Well known examples

- Heyting algebras,
- $[0,1]_G$,
- $[0,1]_{Ł}$ ($x \odot y = \max\{0, x + y - 1\}$)
- $[0,1]_\Pi$ ($\odot = \cdot$)

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$$\mathfrak{M}, v \Vdash p \text{ iff } v \in e(p), \quad \mathfrak{M}, v \Vdash \neg\varphi \text{ iff } v \notin e(\varphi)$$

$$\mathfrak{M}, v \Vdash \varphi\{\wedge,\vee\}\psi \text{ iff } \mathfrak{M}, v \Vdash \varphi \text{ \{and, or\} } \mathfrak{M}, v \Vdash \psi$$

$$\mathfrak{M}, v \Vdash \Box\varphi \text{ iff for all } w \in W \text{ s.t. } R(v,w),\ \mathfrak{M}, w \Vdash \varphi$$

$$\mathfrak{M}, v \Vdash \Diamond\varphi \text{ iff there is } w \in W \text{ s.t. } R(v,w) \text{ and } \mathfrak{M}, w \Vdash \varphi$$

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$$e(v, \neg p) = \neg e(v,p), \qquad e(v, \varphi\{\wedge,\vee\}\psi) = e(v,\varphi)\{\wedge,\vee\}e(v,\psi)$$

$$e(v, \Box\varphi) = \begin{cases} 1 & \text{if for all } w \in W \text{ s.t. } R(v,w),\ e(u,\varphi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

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A residuated lattice.

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safe whenever $e(u, \Box\varphi), e(u, \Diamond\varphi)$ are defined in every world.

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Particularities

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- In (non-modal) MV-logics in general, this D.T already fails. At most weaker versions will be attainable, but still unclear (by semantic methods-only is not easy to see). Over order-preserving logics (eg. $[0,1]_G$) D.T. naturally still holds.

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  - Only very particular cases allow for the above inter-definability of $\Box - \Diamond$ (eg. chains with an involutive negation like $[0, 1]_{Ł}$)
  - (enough) Constants in the language allow certain level of expressability, but as for now, quite ad hoc.

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- In (c.) modal logic $\Diamond$ can be given as an abbreviation of $\Box$ (or vice-versa).
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  - Axioms relating $\Box$ and $\Diamond$ are crucial to get both of them over the same accessibility relation (eg. also intutionistic Modal logics have faced this in different ways)

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- Even in cases where the underlying MV-logic is decidable, the decidability of the MV-modal logics is unclear.

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- This highly complicates the Truth-lemma proof.

Some known results for infinite algebras

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- can we say something else??

From undecidability results...

Undecidable deductions

$\mathcal{A}$ class of linearly ordered R.L such that

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Corollary

$\Vdash^{g}_{\mathcal{M}_{Ł}}$, $\Vdash^{g}_{\mathcal{M}_{\Pi}}$ and $\Vdash^{g}_{\mathcal{M}_{\Pi_1}}$, and their restrictions to finite models are undecidable .

for $\Pi_1 \prec [0,1]_{\Pi}$ with universe $\{0,1\} \cup \{a^i : i \in \omega\}$ with $a \in (0,1)$.

Some hints on the proof...

- Post correspondence problem: given $\langle v_1, w_1 \rangle, \ldots, \langle v_n, w_n \rangle$ of pairs of numbers in some base $s > 1$, it is **undecidable** whether there exist $i_1, \ldots, i_k$ with $i_j \in \{1, \ldots, n\}$ such that $v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k}$.

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Let $P = \{\langle \mathbf{x}_1, \mathbf{y}_1\rangle \ldots \langle \mathbf{x}_n, \mathbf{y}_n\rangle\}$. Define Γ_P over $\mathcal{V} = \{x, y, z\}$ as

$$\neg\Box 0 \to (\Box p \leftrightarrow \Diamond p) \text{ for each } p \in \mathcal{V},$$

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and $\varphi_P = (x \leftrightarrow y) \to (z \vee (x \to xz))$.

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Theorem

$$P \text{ is SAT} \iff \Gamma_P \not\Vdash^g_{\mathcal{M}_\mathcal{A}} \varphi_P \iff \Gamma_P \not\Vdash^g_{\omega\mathcal{M}_\mathcal{A}} \varphi_P$$

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- The $\Leftarrow$ direction uses weakly saturation and non-contractivity to prove that if $\Gamma_P \not\Vdash_{\mathcal{K}} \varphi_P$ then it happens in a model with structure as above with an evaluation that is then easily translatable into a solution of P.

...to (non) RE logics

In general

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If $\models_{\mathcal{C}}$ is decidable, then $\not\Vdash^{g}_{\omega\mathcal{M}_{\mathcal{C}}}$ is recursively enumerable.

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If $\mathcal{C}$ of R.L is as in Lemma $(\star)$ and $\models_{\mathcal{C}}$ is decidable, then $\Vdash^{g}_{\omega\mathcal{M}_{\mathcal{C}}}$ is not R.E, and so, not axiomatizable.

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However, it is not the case that $\Vdash^{g}_{\omega\mathcal{M}_{Ł}} = \Vdash^{g}_{\mathcal{M}_{Ł}}$, nor for the product case ...

so what about $\Vdash^{g}_{\mathcal{M}_{Ł}}$ and $\Vdash^{g}_{\mathcal{M}_{\Pi}}$? (the modal Łukasiewicz/product logics?)

The Łukasiewicz case

A model $\mathfrak{M}$ is witnessed iff for all $v \in W$, φ, there are $w_{\Box\varphi}$, $w_{\Diamond\varphi}$

$$e(v, \Box\varphi) = e(w_{\Box\varphi}, \varphi) \quad \text{and} \quad e(v, \Diamond\varphi) = e(w_{\Diamond\varphi}, \varphi)$$

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We have completeness wrt. finite-width models... but the depth might still be infinite

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Lemma

$\Gamma \Vdash^{g}_{\omega\mathcal{M}_{Ł}} \varphi$ iff $\Gamma, \Upsilon(p,q) \Vdash^{g}_{\mathcal{M}_{Ł}} \varphi \vee \Psi(p,q)$ for any $p, q \notin \mathcal{V}ars(\Gamma, \varphi)$ and

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Theorem

The finitary companion of $\Vdash^{g}_{\mathcal{M}_{Ł}}$ is not RE.

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! not known anything like the completeness of $\Vdash^{g}_{\mathcal{M}_{\Pi}}$ wrt witnessed models (only a partial result, not generalizable, for theorems).

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Gödel modal logics

Differentiating underlying algebras

Let $G_{\downarrow} := \{0\} \cup \{1/i \colon i \in N^*\}$.

Lemma

$\Gamma \vdash_{[0,1]_G} \varphi$ iff $\Gamma \vdash_{G_{\downarrow}} \varphi$

Differentiating underlying algebras

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Theorem (Hajek, 2005; Baaz, 1995)

$\vdash_{FOG_{\downarrow}}$ is non-arithmetical.
The $\exists$-free fragment is not recursively enumerable.

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Theorem

$K(G)_{\Box}$ (Caicedo and Rodríguez (2010)) +
$((\Box\varphi \leftrightarrow \Box\psi) \wedge (\Box(\varphi \to \psi) \to \varphi)) \to (\Box\psi \vee \neg\Box\psi)$ is complete wrt.
$\Diamond$-free fragment over $G_{\downarrow}$.

Thank you!

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