Gaps between cardinalities of quotient algebras of rank-into-rank embeddings

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Topics

Complex analysis

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Algebra-Self distributivity

Set theory-Rank-into-rank cardinals

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Large cardinal hierarchy

1. Rank-into-rank embeddings
2. -n-huge
3. Huge
4. Vopenka
5. Extendible
6. Supercompact
7. Woodin
8. Measurable
9. Ramsey
10. Weakly compact
11. Inaccessible cardinals
12. ZFC

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For $n \in \omega$, define non-commutative polynomials $p^*_{n,j_1,\dots,j_k}(x_1,\dots,x_k)$ by letting

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$$= 1 + \sum\{x_{a_1}\dots x_{a_s} \mid \mathrm{crit}(j_{a_1} * \dots * j_{a_s}) = \mathrm{crit}_n(j_1,\dots,j_k),$$

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Let $\mathcal{A}$ be the set of all functions $f : \omega \setminus \{0\} \to \mathbb{C}$. Define the Dirichlet convolution $*$ on $\mathcal{A}$ by letting $x * y(\ell) = \sum_{j \cdot k = \ell} x(j) \cdot y(k)$. Then $(\mathcal{A}, +, *)$ is a ring. The ring $\mathcal{A}$ has identity e defined by $e(k) = 0$ for $k > 1$ and $e(1) = 1$. If $x \in \mathcal{A}, x(1) \neq 0$, then x is invertible.

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Define $1 \in \mathcal{A}$ by letting $1(n) = 1$ for all n. Define $\mu \in \mathcal{A}$ by letting $\mu(n^2 x) = 0$ whenever $n > 1$ and $\mu(p_1 \dots p_k) = (-1)^k$ whenever $p_1, \dots, p_k$ are distinct primes.

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If u is a function holomorphic on a neighborhood of 0 with $u(0) = 1$, then define $R_u, r_u \in \mathcal{A}$ by letting

$$\ln(u(z)) = \sum_{j=1}^{\infty} R_u(j)z^j;\ r_u = T(R_u);\ \frac{zu'(z)}{u(z)} = \sum_{k=1}^{\infty} r_u(k)x^k.$$

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Suppose that u is a function holomorphic on a neighborhood of 0 with $u(0)=1$, $a \in \mathcal{A}$ and

$$\prod_{k=1}^{\infty}(1+z^k)^{a(k)} = \frac{1}{u(z)}$$

for z in a neighborhood of 0. Then

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Bounds from infinite products of polynomials

Define $p^*_{n,j_1,\dots,j_k}(x) = p^*_{n,j_1,\dots,j_k}(x,\dots,x)$.

Theorem

(V.) Suppose that $j_1,\dots,j_k \in \mathcal{E}^+_\lambda$. *Let* $X = \langle j_1,\dots,j_k\rangle$, *and let* $\alpha_n = \mathrm{crit}_n(j_1,\dots,j_k)$ *for* $n \in \omega$. *Suppose that* N *is a natural number. For each natural number* h,

1. *let* $b(h)$ *be the number of* $n \geq N$ *such that* $|X/\equiv^{\alpha_{n+1}}| - |X/\equiv^{\alpha_n}| = h$, *and*
2. *let* $c(h)$ *be the number of* $n \geq N$ *such that* $\mathrm{ht}(X/\equiv^{\alpha_{n+1}}) - \mathrm{ht}(X/\equiv^{\alpha_n}) = h$.

Let $u(x) = (1-kx)\cdot p^*_{0,j_1,\dots,j_k}(x)\dots p^*_{N-1,j_1,\dots,j_k}(x)$. *Then*

1. *Either* $b = T^{-1}(r_u * \mu^\sharp)$ *or there is some* $h \geq 1$ *with* $b(h) < T^{-1}(r_u * \mu^\sharp)(h)$.
2. *Either* $c = T^{-1}(r_u * \mu^\sharp)$ *or there is some* $h \geq 1$ *with* $c(h) < T^{-1}(r_u * \mu^\sharp)(h)$.

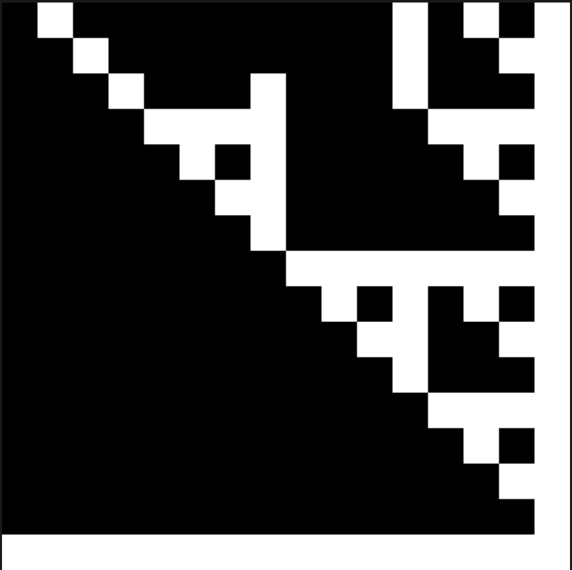
An algebraization of $\mathcal{E}_\lambda$.

A reduced Laver-like algebra is an algebra $(X, *, 1)$ that satisfies the identities $x*(y*z) = (x*y)*(x*z), x*1 = 1, 1*x = x$ and where if $x_n \in X$ for $n \in \omega$, then there is some N with $x_0 * \cdots * x_N = 1$.

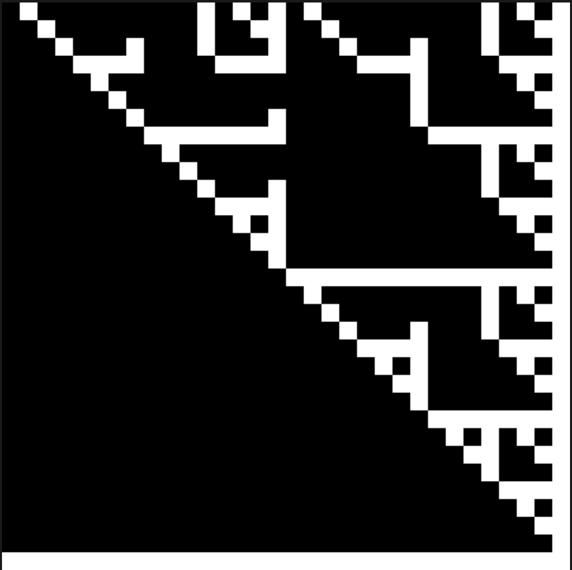
The algebras $(\mathcal{E}_\lambda / \equiv^\gamma, *)$ are always reduced Laver-like algebras, but there are many reduced Laver-like algebras that do not embed into any $(\mathcal{E}_\lambda / \equiv^\gamma, *)$. The main results of this talk could have been stated in greater generality in terms of Laver-like algebras.

Computer calculations give some evidence that the bounds on b, c are sharp, or nearly sharp, and at the very least difficult to improve upon but only when these bounds are generalized to the setting of reduced Laver-like algebras.

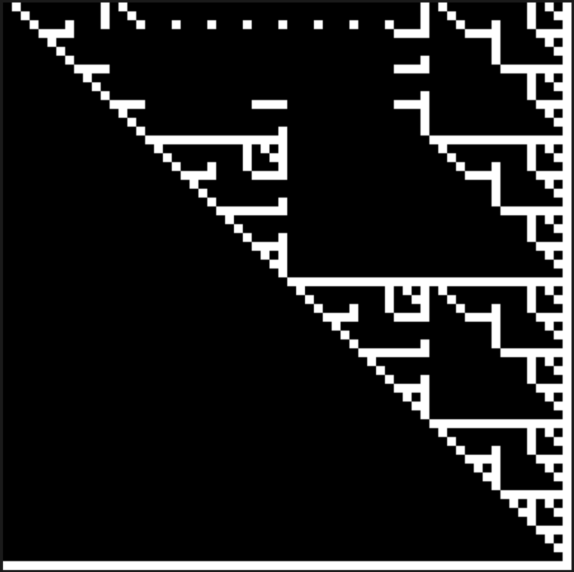
Photographic evidence of rank-into-rank cardinals: A_4



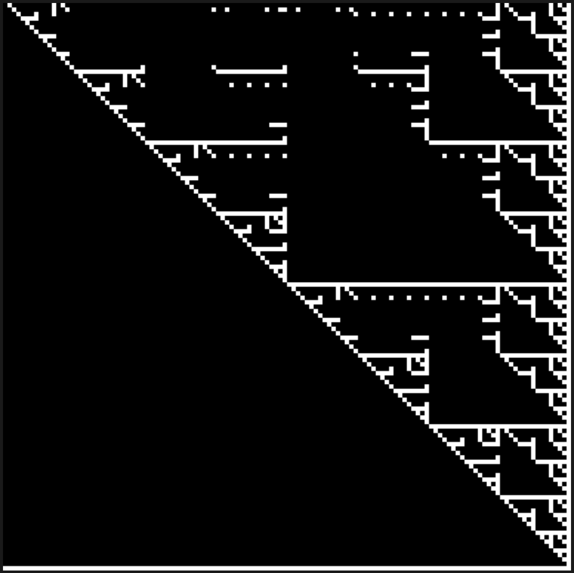
Photographic evidence of rank-into-rank cardinals: A_5



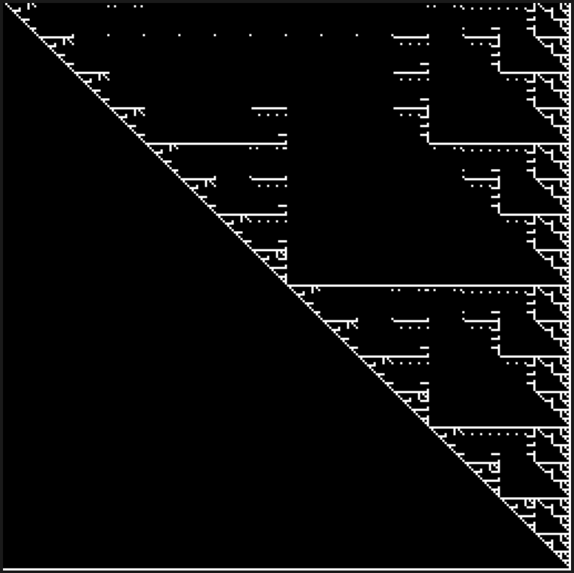
Photographic evidence of rank-into-rank cardinals: A_6



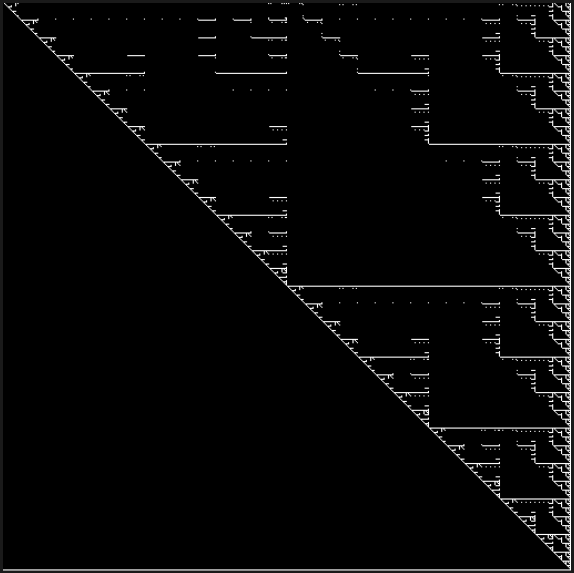
Photographic evidence of rank-into-rank cardinals: A_7



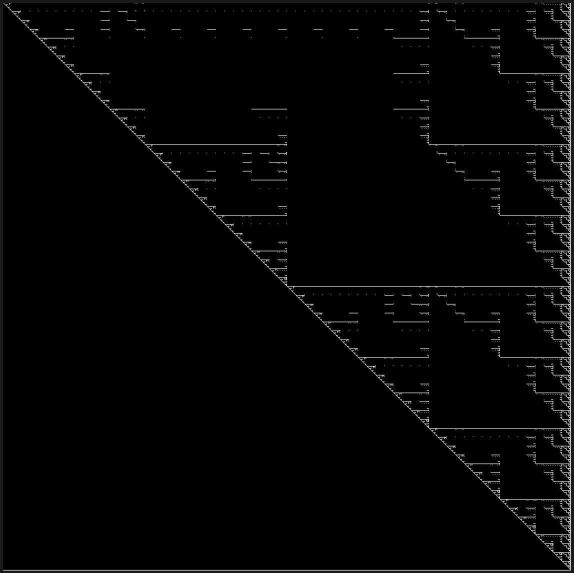
Photographic evidence of rank-into-rank cardinals: A_8



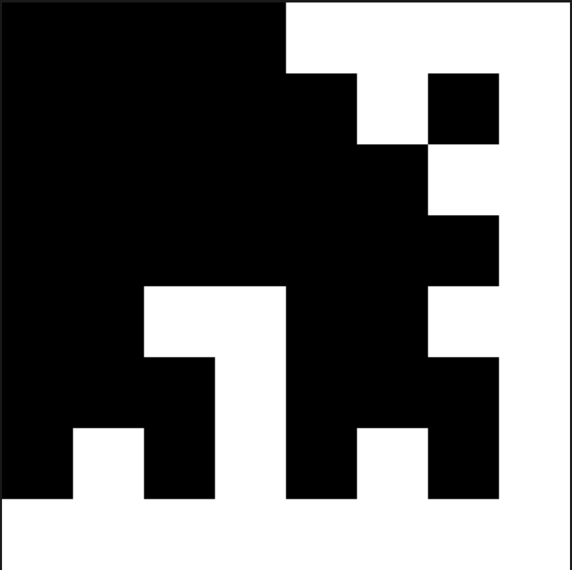
Photographic evidence of rank-into-rank cardinals: A_9



Photographic evidence of rank-into-rank cardinals: A_{10}

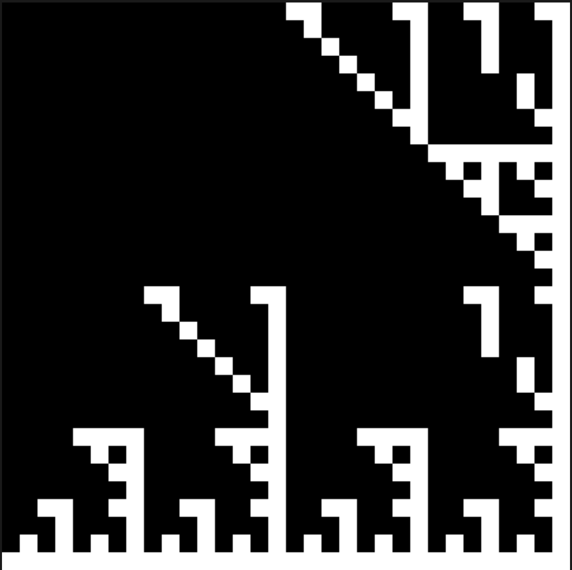


Photographic evidence of rank-into-rank cardinals: A_3



Photographic evidence of rank-into-rank cardinals: A_4

Photographic evidence of rank-into-rank cardinals: A_5



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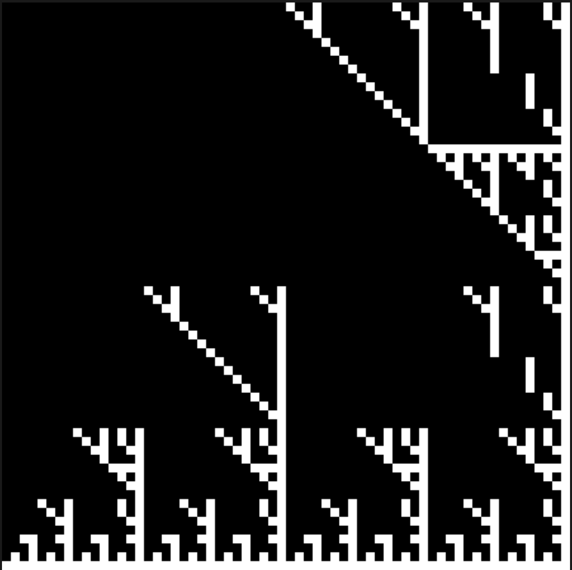

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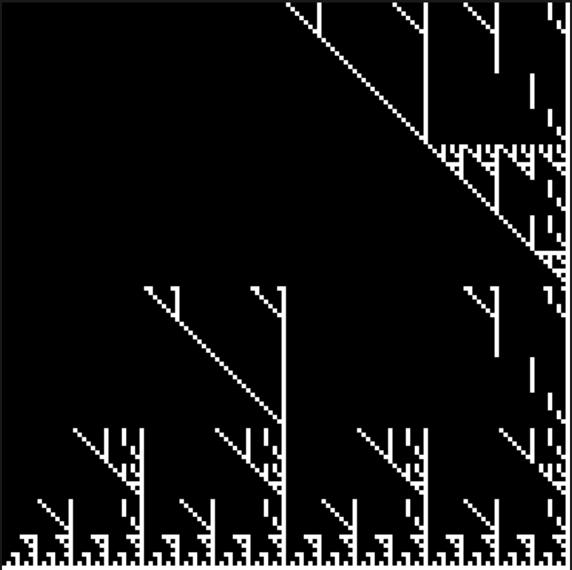
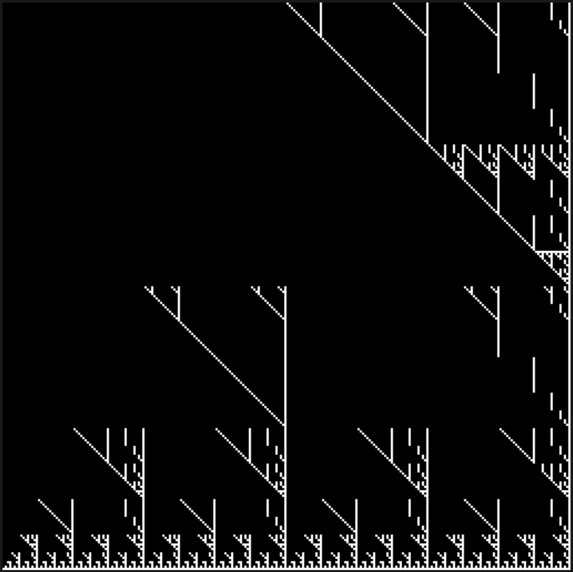
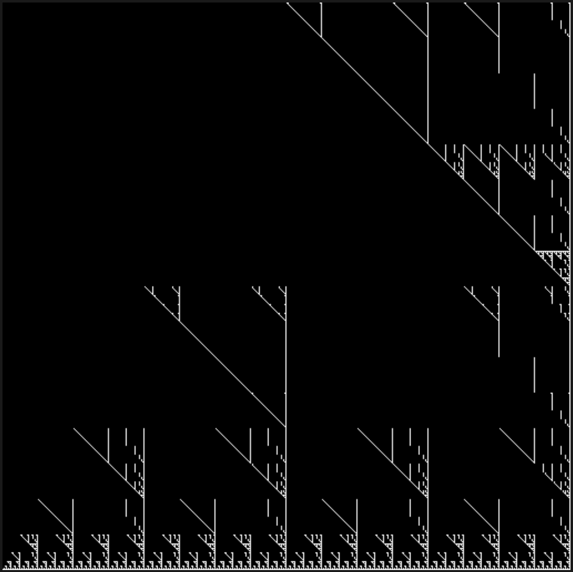


Photographic evidence of rank-into-rank cardinals: A_8



Photographic evidence of rank-into-rank cardinals: A_9



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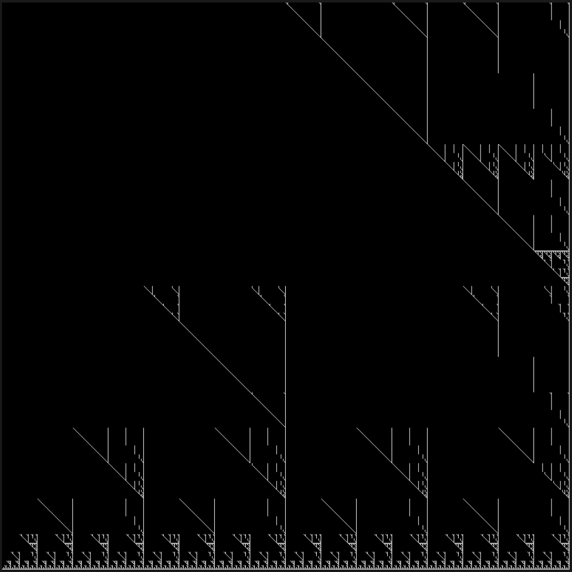
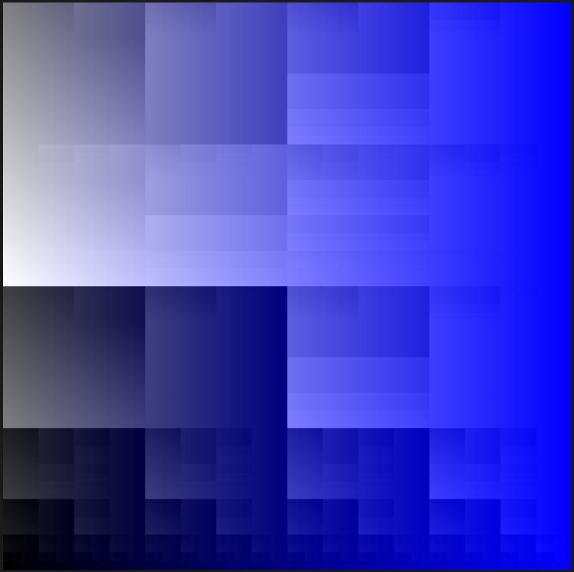
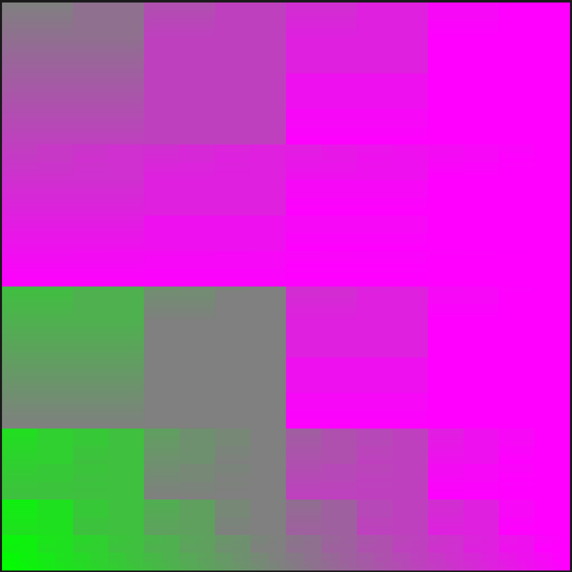


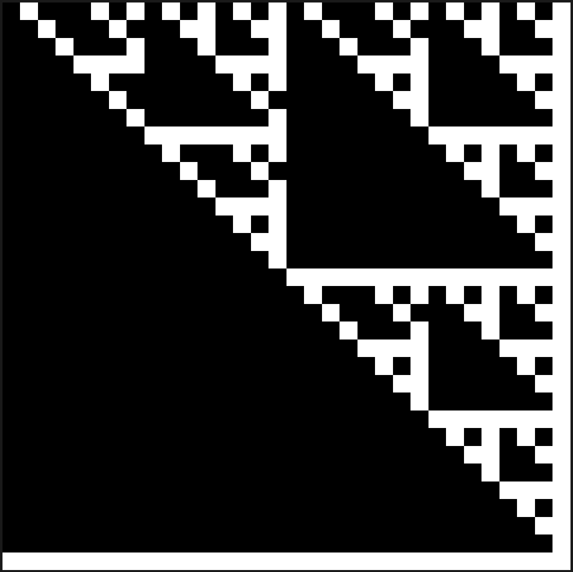
Heat map



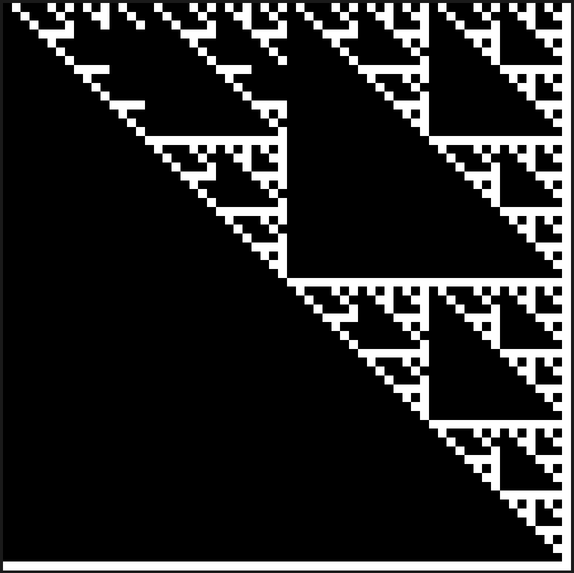
Heat map



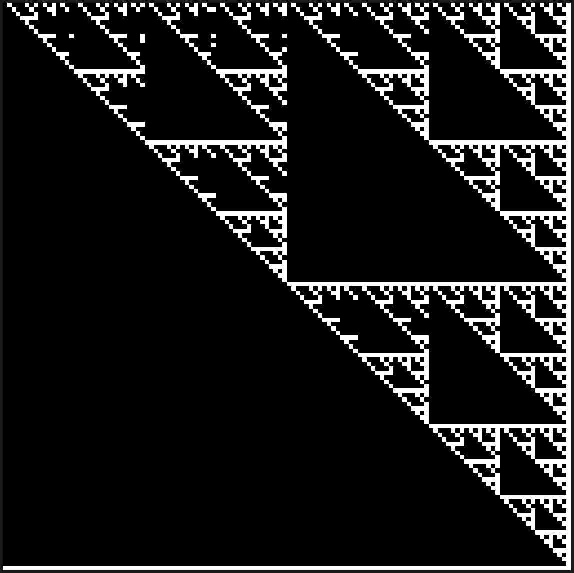
Pictures: FM_5^-



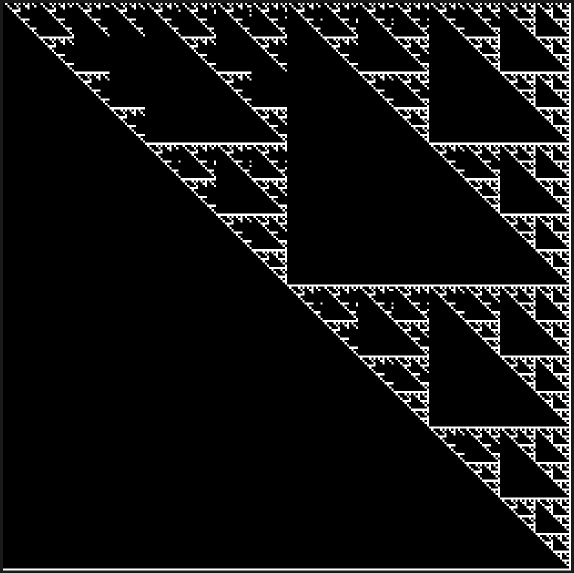
Pictures: FM_6^-



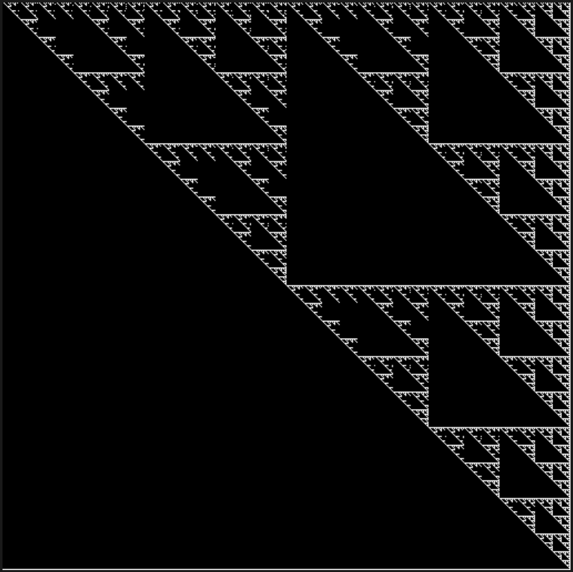
Pictures: FM_7^-



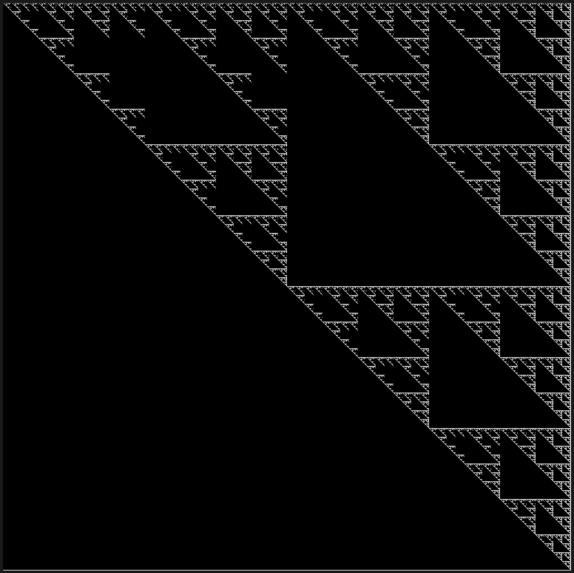
Pictures: FM_8^-



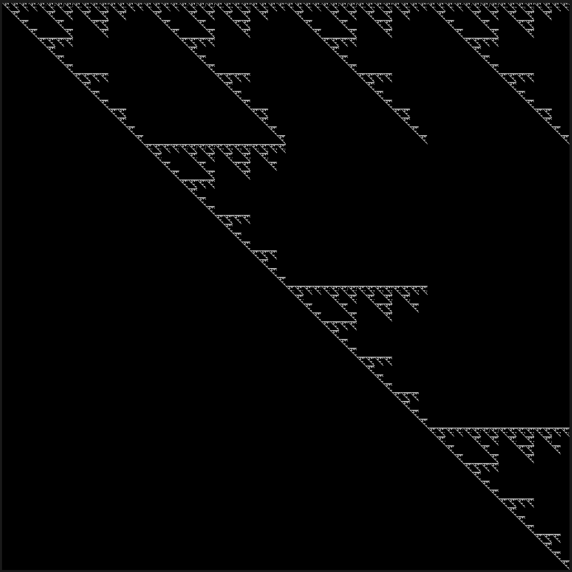
Pictures: FM_9^-



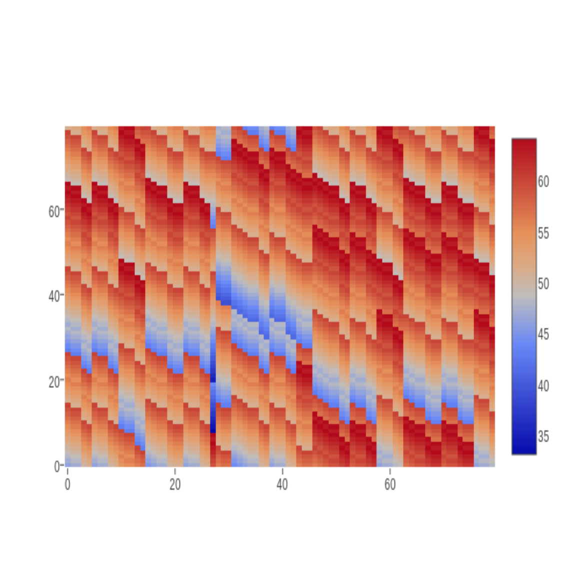
Pictures: FM_{10}^{-}



Pictures:Multigenic Laver table snapshot

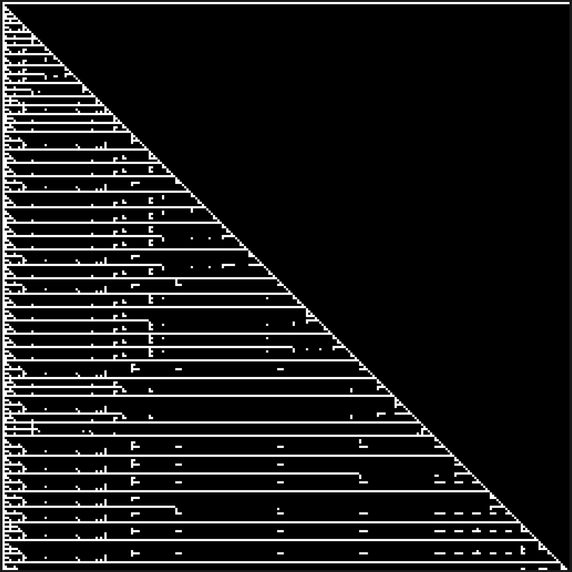


Pictures:Endomorphic Laver table snapshot



Pictures:Endomorphic Laver table snapshot

Other table (truncated)



The end: Happy Bitcoin Pizza Day!