Gluing residuated lattices

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Ongoing joint work with Nick Galatos

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A residuated lattice, or RL, is a structure $\mathbf{A} = (A, \cdot, \backslash, /, \wedge, \vee, 1)$ where:

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a monoid,
- $x \cdot y \leq z$ iff $y \leq x\backslash z$ iff $x \leq z/y$.

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We call a residuated lattice integral if 1 is the top element of the lattice, and bounded if there is an extra constant 0 in the signature that is the least element of the lattice.

The RL is commutative if the monoid is commutative, and we write $x \to y$ for $x\backslash y = y/x$.

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Residuated lattices are the equivalent algebraic semantics of substructural logics, that include: classical logic, intuitionistic logic, linear logic, relevance logic, fuzzy logics...

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  For $I, J \in \mathcal{I}(\mathbf{R})$ define

$$I \wedge J = I \cap J, \quad I \vee J = \{a + b : a \in I \text{ and } b \in J\}$$

$$I \cdot J = \{a_1 b_1 + \ldots + a_n b_n : a_i \in I, b_i \in J\}$$

$$I \backslash J = \{a \in R : Ia \subseteq J\}, \quad J/I = \{a \in R : aI \subseteq J\}$$

$(\mathcal{I}(\mathbf{R}), \wedge, \vee, \cdot, \backslash, /, R)$ is a residuated lattice.

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- $\mathbf{Z}^- = \{\mathbb{Z}^-, +, \ominus, \min, \max, 0\}$, where $\ominus$ is the truncated difference, is a residuated lattice.

- Residuated lattices do not satisfy any special purely lattice-theoretic or monoid-theoretic property

- R. Belohlavek, V. Vychodil, *Residuated Lattices of Size* ≤ 12, Order, 27:147–161, 2010 :

	1	2	3	4	5	6	7	8	9	10	11	12
Lattices	1	1	1	2	5	15	53	222	1,078	5,994	37,622	262,776
Residuated lattices	1	1	2	7	26	129	723	4,712	34,698	290,565	2,779,183	30,653,419
Linear res. lattices	1	1	2	6	22	94	451	2,386	13,775	86,417	590,489	4,446,029

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- Large classes of residuated lattices currently lack a structural understanding.

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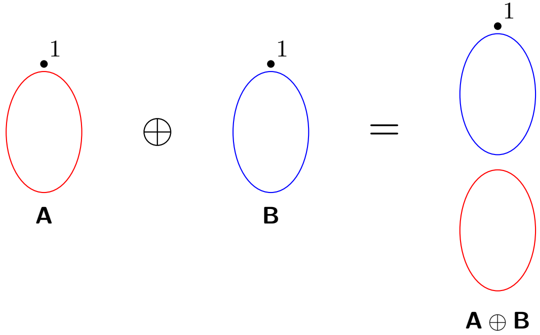

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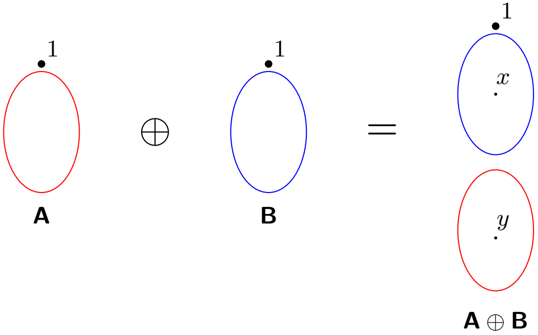

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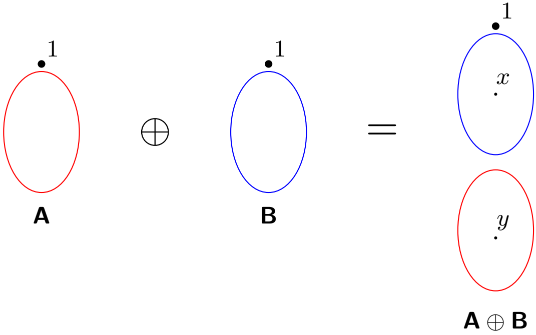

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Ordinal sum

The ordinal sum has proved to be a powerful construction, in particular in the realm of the algebraic semantics of fuzzy logics.

We call basic hoops semilinear (subdirect products of chains) commutative integral residuated lattices satisfying divisibility: $x \wedge y = x \cdot (x \to y)$, and BL-algebras are bounded basic hoops. BL-algebras are the algebraic semantics of Hájek's Basic Logic, the logic of continuous t-norms.

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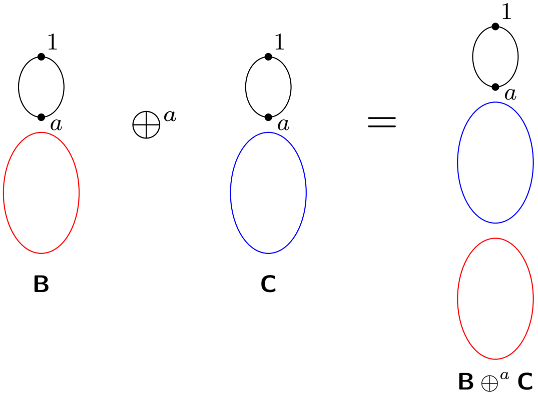
The ordinal sum construction intuitively stacks one (semi)lattice on top of the other gluing together the top elements.

What if we glue more? For example, a filter.

Let $\mathbf{B} = (B, \cdot, \backslash, /, \wedge, 1)$, $\mathbf{C} = (C, \cdot, \backslash, /, \wedge, 1)$ be integral residuated $\wedge$-semilattices, that intersect in a principal lattice filter generated by an element a that is conical (comparable with all other elements) and idempotent $(a \cdot a = a)$.



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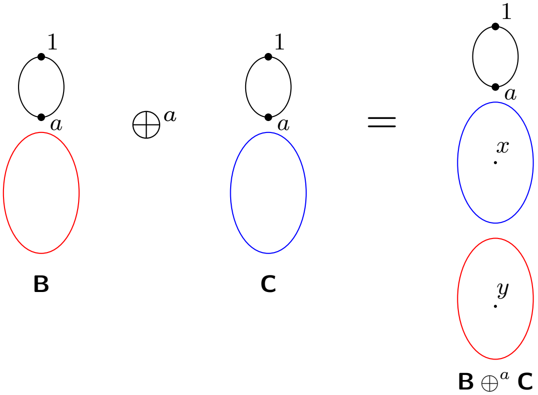

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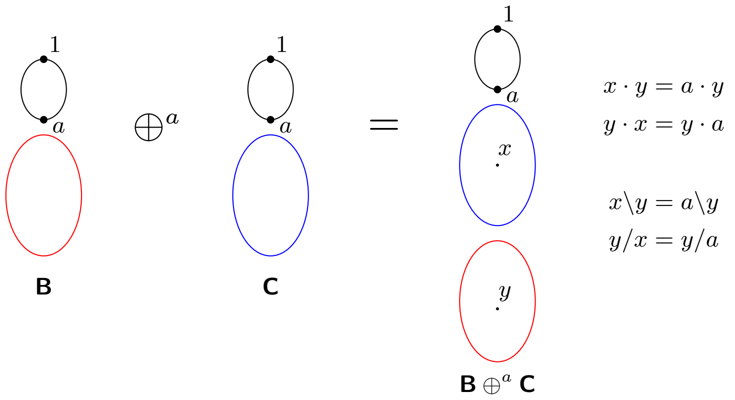

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Properties

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- all unary equations without join (e.g. it preserves n-potency, $x^n = x^{n+1}$, for every $n \geq 1$)
- If a is join irreducible in **B**, any gluing of the kind $\mathbf{B} \oplus_a \mathbf{C}$ preserves all unary equations satisfied by both **B** and **C**.

Filters

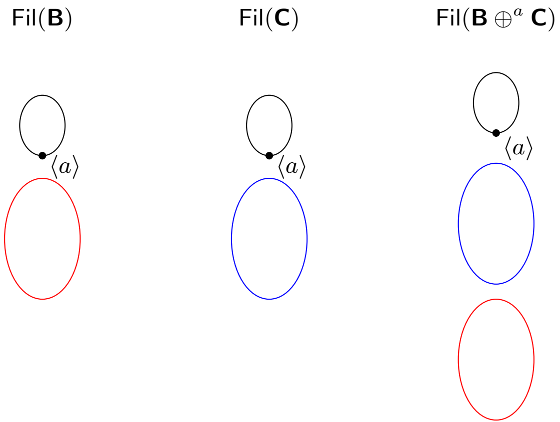
Let us consider now gluings of commutative integral residuated lattices (CIRLs).

Given a CIRL **A**, $F \subseteq A$ is a deductive filter (or just filter) if it is a lattice filter closed under product.

The lattice of deductive filters is dually isomorphic to the congruence lattice via the maps: $F \mapsto \theta_F = \{(x, y) \in A : x \to y, y \to x \in F\}$ and $\theta \mapsto F_\theta = \{a : (a, 1) \in \theta\}$.

Since a is an idempotent element, the lattice filter and the deductive filter generated by a coincide.

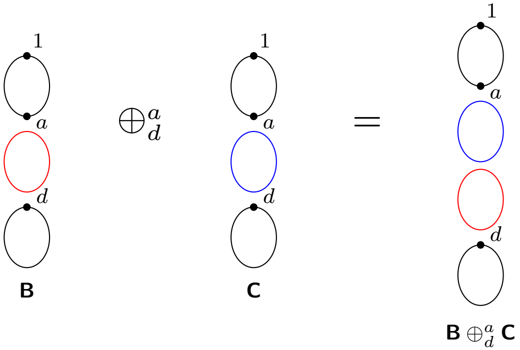
In fact, the gluing construction is reflected in the deductive filters lattice.



Since a coatom in the lattice of deductive filters would need to be above $\langle a \rangle$, $\mathbf{B} \oplus^a \mathbf{C}$ is subdirectly irreducible iff $\mathbf{B}$ and $\mathbf{C}$ are subdirectly irreducible.

What if we want to glue a filter and an ideal?

Let now **B** $= (B, \cdot, \backslash, /, \wedge, 1)$, **C** $= (C, \cdot, \backslash, /, \wedge, 1)$ be integral residuated $\wedge$-semilattices, $B \cap C = \uparrow a \cup \downarrow d$ with a, d conical and idempotent.



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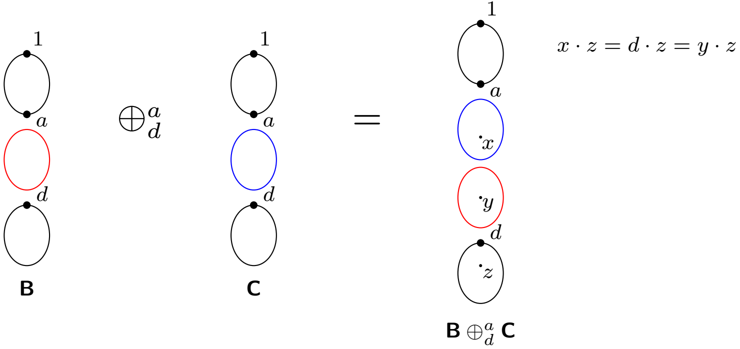

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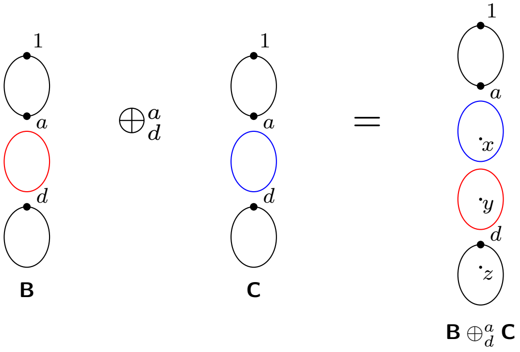

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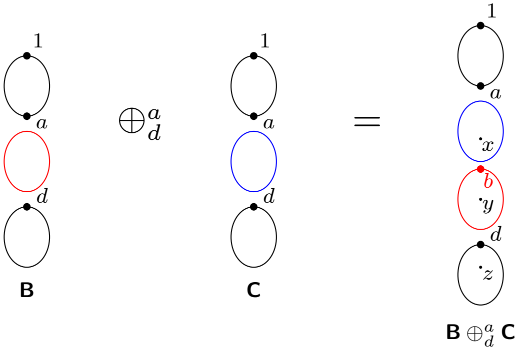

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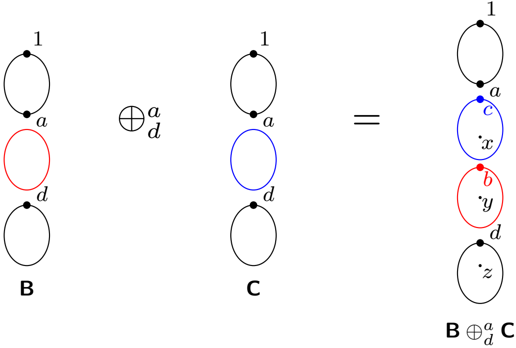

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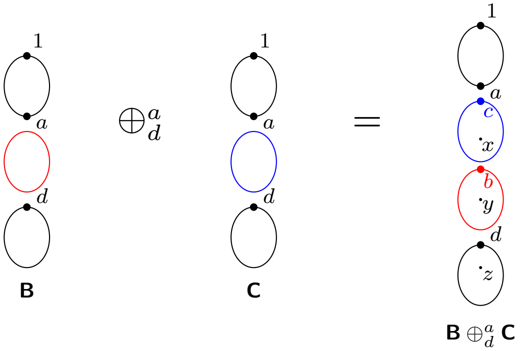

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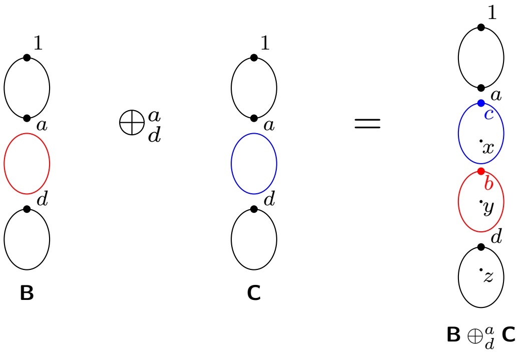

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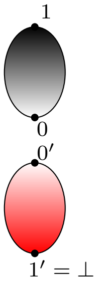
Properties

This new gluing construction still preserves commutativity and prelinearity, but it does not preserve divisibility.

In CIRLs, since a is again an idempotent conical element, gluings of subdirectly irreducible are subdirectly irreducible.

Examples

Given any bounded commutative residuated lattice **A**, we can glue its disconnected rotation and its n-lifting: $Ł_n \oplus \mathbf{A}$



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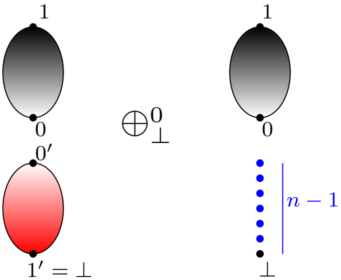

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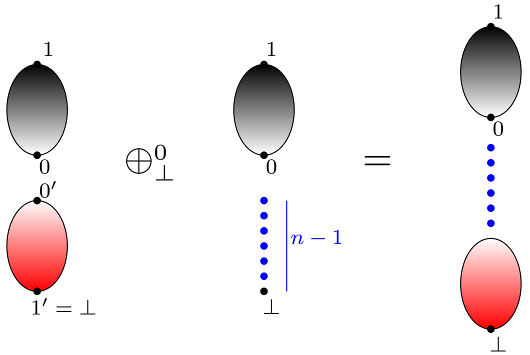

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Examples

Let us consider bounded totally ordered CIRLs that are 2-potent, i.e. that satisfy $x^2 = x^3$, and such that for every x, y

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This is a positive universal first-order formula, these structures generate a variety of MTL-algebras (bounded semilinear CIRLs) that satisfy $x^2 = x^3$ and

$$x \vee \ ((x \cdot (x \wedge y)) \backslash (x \wedge y)^2) = 1.$$

This variety has the FMP (finite model property, i.e. it is generated by its finite members).

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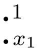

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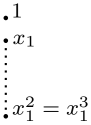

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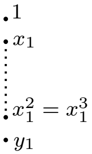

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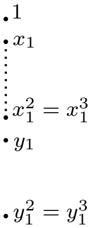

Example

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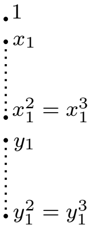

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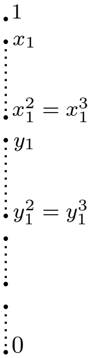

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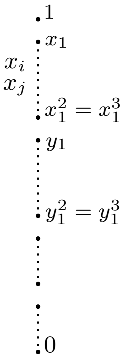

$$x_i \cdot x_j = x_i^2 = x_j^2 = x_1^2$$

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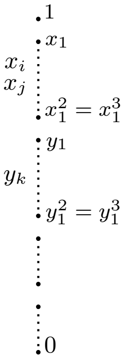

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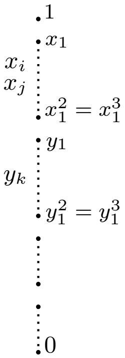

$$x_i \cdot x_j = x_i^2 = x_j^2 = x_1^2$$

$$x_i \cdot y_k \leq y_k^2 = y_1^2 \;\Rightarrow\; x_i \cdot y_k = y_1^2 = x_1^2 \cdot y_k$$

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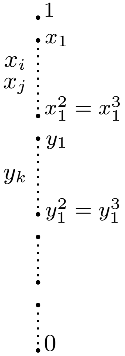


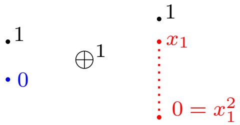
$$x_i \cdot x_j = x_i^2 = x_j^2 = x_1^2$$

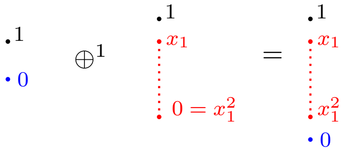
$$x_i \cdot y_k \leq y_k^2 = y_1^2 \;\Rightarrow\; x_i \cdot y_k = y_1^2 = x_1^2 \cdot y_k$$

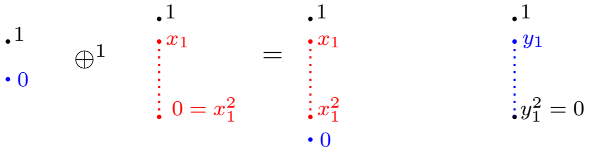
We can generate these chains as gluings of finite simple 2-potent MTL-chains.

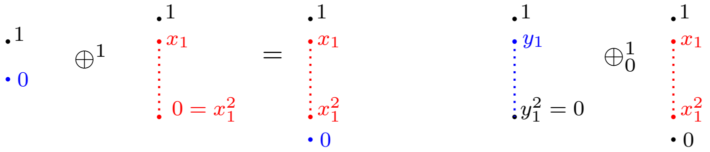
•1

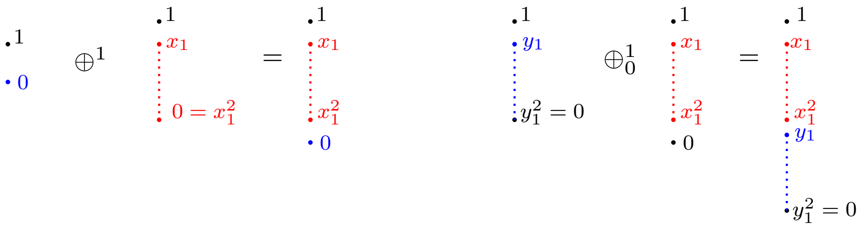
• 0

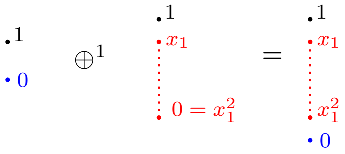

1
0
$\oplus^1$
1
x_1
$0 = x_1^2$

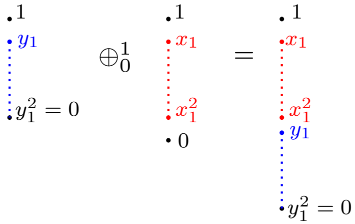

1
0
$\oplus^1$
1
x_1
$0 = x_1^2$
=
1
x_1
x_1^2
0

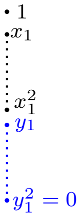

1
0
⊕1
1
x_1
$0 = x_1^2$
=
1
x_1
x_1^2
0
1
y_1
$y_1^2 = 0$

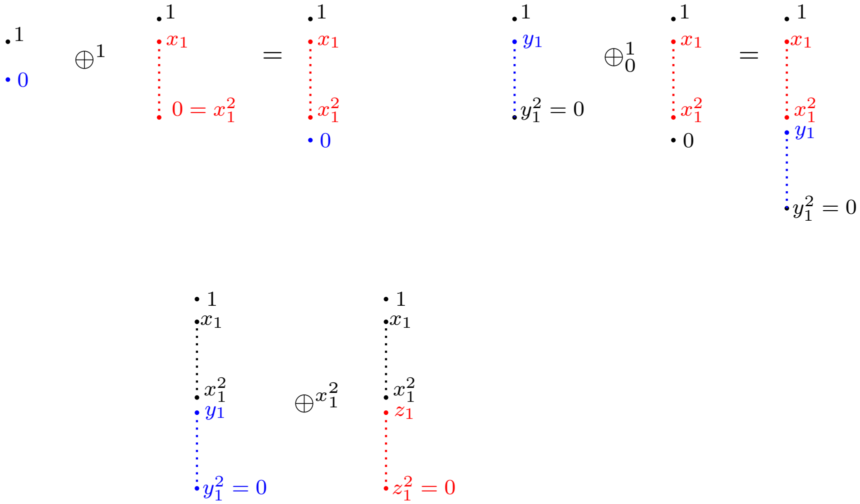

1
0
$\oplus^1$
1
x_1
$0 = x_1^2$
=
1
x_1
x_1^2
0
1
y_1
$y_1^2 = 0$
$\oplus_0^1$
1
x_1
x_1^2
0

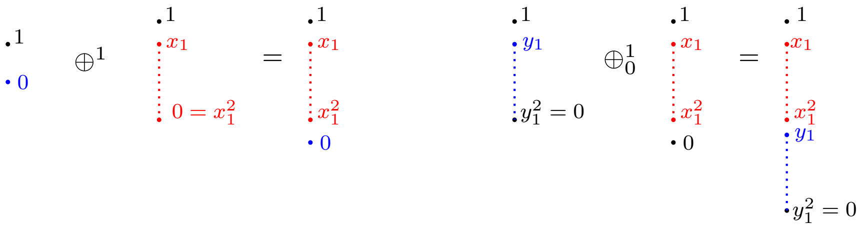

1
0
⊕1
1
x1
0 = x1^2
=
1
x1
x1^2
0
1
y1
y1^2 = 0
⊕1
0
1
x1
x1^2
0
=
1
x1
x1^2
y1
y1^2 = 0

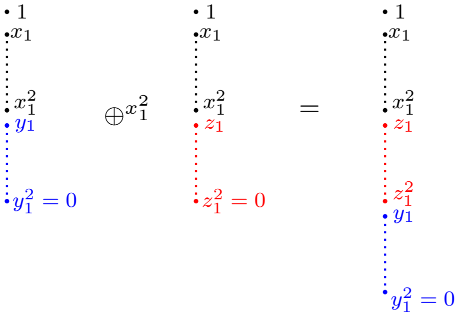

1
0
$\oplus^1$
1
x_1
$0 = x_1^2$
=
1
x_1
x_1^2
0


1
y_1
$y_1^2 = 0$
$\oplus_0^1$
1
x_1
x_1^2
0
=
1
x_1
x_1^2
y_1
$y_1^2 = 0$


1
x_1
x_1^2
y_1
$y_1^2 = 0$


1
0
$\oplus^1$
1
x_1
$0 = x_1^2$
=
1
x_1
x_1^2
0
1
y_1
$y_1^2 = 0$
$\oplus_0^1$
1
x_1
x_1^2
0
=
1
x_1
x_1^2
y_1
$y_1^2 = 0$
1
x_1
x_1^2
y_1
$y_1^2 = 0$
$\oplus^{x_1^2}$
1
x_1
x_1^2
z_1
$z_1^2 = 0$


1
0
⊕¹
1
x_1
$0 = x_1^2$
=
1
x_1
x_1^2
0
1
y_1
$y_1^2 = 0$
$\oplus_0^1$
1
x_1
x_1^2
0
=
1
x_1
x_1^2
y_1
$y_1^2 = 0$


1
x_1
x_1^2
y_1
$y_1^2 = 0$
$\oplus^{x_1^2}$
1
x_1
x_1^2
z_1
$z_1^2 = 0$
=
1
x_1
x_1^2
z_1
z_1^2
y_1
$y_1^2 = 0$

Thank you!