Subvariety containment for idempotent semirings

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Residuated Lattices

A (commutative) **residuated lattice** is an algebraic structure $\mathbf{R} = (R, \vee, \wedge, \cdot, \backslash, /, 1)$, such that

- $(R, \vee, \wedge)$ is a lattice
- $(R, \cdot, 1)$ is a (commutative) monoid
- For all $x, y, z \in R$

  $$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y,$$

  where $\leq$ is the induced lattice order.

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- (C)RL denotes the **variety** of (commutative) residuated lattices.
- multiplication is order preserving:

$$x \leq y \implies zx \leq zy \quad \& \quad xz \leq yz$$

- multiplication distributes of join:

$$x(y \vee z) = xy \vee xz \quad \& \quad (y \vee z)x = yx \vee zx$$

Residuated structures are the algebraic semantics of substructural logics (i.e., axiomatic extension of the **Full Lambek Calculus**) $\mathbf{FL}$.

Structural rules have an algebraic meaning.

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Pi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \Pi} \text{ (e)} \quad \Leftrightarrow \quad xy \leq yx$$

$$\frac{\Gamma, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi} \text{ (w)} \quad \Leftrightarrow \quad x \leq 1$$

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi} \text{ (c)} \quad \Leftrightarrow \quad x \leq x^2$$

We can use algebraic methods to answer questions about the logics.

ISR and ISR$_\perp$

The $\{\vee, \cdot, 1\}$-fragment of RL is an idempotent semiring (ISR).

Definition

An **idempotent semiring** is an algebra $\mathbf{S} = (S, \vee, \cdot, 1)$ where:

- $(S, \vee)$ is an idempotent semigroup
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An algebra $\mathbf{S} = (S, \vee, \cdot, \bot, 1)$ is an **idempotent semiring with** $\bot$ if $(S, \vee, \cdot, 1)$ is an ISR where additionally $(S, \vee, \bot)$ is monoid and $\bot x = x\bot = \bot$ for all $x \in S$.

We denote the variety of idempotent semirings (with $\bot$) by ISR (ISR$_\bot$).

Decidability and the $\{\vee, \cdot, 1\}$-fragment

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- In (S. PhD Thesis [2019]), we prove undecidability of the (Eq. Th) word problem for any variety in the interval CRL $+ \Sigma$ to RL, where Σ is any (**expansive**) **non-spinal equations** finite set of $\{\vee, \cdot, 1\}$-equations.

For sets of $\{\vee, \cdot, 1\}$-equations Γ, Σ, we want to know:

$$\mathsf{RL} + \Gamma \models \Sigma?$$

Or equivalently:

$$\mathsf{RL} + \Gamma \subseteq \mathsf{RL} + \Sigma?$$

Equations in the signature $\{\vee, \cdot, 1\}$

We call an equation $s = t$ in the signature $\{\vee, \cdot, 1\}$ is ISR-equivalent to a **basic equation**

$$[A]: \quad a_0 \leq \bigvee_{a \in A} a$$

where

- a is a monoid term
- A is a finite nonempty set of monoid terms
- We associate $[A]$ with the pair (a_0, A)

Simple equations

Definition

Let $[A] = (a_0, A)$ be a basic equation

$$[A] : \quad a_0 \leq \bigvee_{a \in A} a.$$

We say $[A]$ is:

- **Linear** if a_0 is linear, i.e., $a_0 = x_1 \cdots x_n$ for some $n > 1$.
- **Proper** if all variables present in A are present in a_0.
- **Simple** if $[A]$ is proper and linear.
- **Degenerate** if for each $a \in A$ there appears a variable not appearing in a_0.

Linearization

Proposition

The following hold:

- *In* ISR*, every basic equation is equivalent to a linear equation.*
- *In* RL *and* $\mathsf{ISR}_\perp$*, every non-degenerate equation is equivalent to a simple equation.*
- *In* RL *and* $\mathsf{ISR}_\perp$*, every degenerate equation is equivalent to* $1 \leq x$*.*

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Such conjoins can be determined by the properties of ISR by **linearization**

$$(\forall u)(\forall v)\ u^2 v \le u^3 \vee uv$$

is equivalent to, via the substitution σ: $u \overset{\sigma}{\mapsto} x \vee y$ and $v \overset{\sigma}{\mapsto} z$,

$$(\forall x)(\forall y)(\forall z)\ xyz \le x^3 \vee x^2 y \vee xy^2 \vee y^3 \vee xz \vee yz$$

Simple Equations and Simple Rules

Any simple equation $[\mathrm{R}]$ corresponds to a **simple structural rule** (R). For example

$$[\mathrm{R}] : xy \leq x^2 \vee y \iff \frac{\Delta_1, \Gamma, \Gamma, \Delta_2 \Rightarrow \Pi \quad \Delta_1, \Psi, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \Psi, \Delta_2 \Rightarrow \Pi} \ (\mathrm{R})$$

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In general,

$$[\mathrm{R}] : x_1 \cdots x_n \leq \bigvee_{r \in \mathrm{R}} r \iff \frac{\{\Delta_1, r^{\mathbf{FL}}(\Gamma_1, \ldots, \Gamma_n), \Delta_2 \Rightarrow \Pi\}_{r \in \mathrm{R}}}{\Delta_1, \Gamma_1, \ldots, \Gamma_n, \Delta_2 \Rightarrow \Pi} \; (\mathrm{R})$$

Theorem [Galatos & Jipsen 2013]

Extensions of $\mathbf{FL}$ by simple rules enjoy **cut-elimination**.

Residuated frames

Definition

A **residuated frame** is a structure $\mathbf{W} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/, 1)$, s.t.

- $(W, \circ, 1)$ is a monoid and W' is a set.
- $N \subseteq W \times W'$,
- $\backslash\!\!\backslash : W \times W' \to W'$ and $/\!\!/ : W' \times W \to W'$ such that
- N is **nuclear**, i.e. for all $u, v \in W$ and $w \in W'$,
$$(u \circ v)\; N\; w \quad \text{iff} \quad u\; N\; (w /\!\!/ v) \quad \text{iff} \quad v\; N\; (u \backslash\!\!\backslash w).$$

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$$(u \circ v)\ N\ w \quad \text{iff} \quad u\ N\ (w /\!/ v) \quad \text{iff} \quad v\ N\ (u \backslash\!\backslash w).$$

$$\wp(W) \underset{\triangleleft}{\overset{\triangleright}{\rightleftarrows}} \wp(W') : \quad \begin{aligned} X^{\triangleright} &= \{y \in W' : X\ N\ y\} \\ Y^{\triangleleft} &= \{x \in W : x\ N\ Y\} \end{aligned}$$

- $({}^{\triangleright}, {}^{\triangleleft})$ is a Galois connection.
- The map $X \xmapsto{\gamma_N} X^{\triangleright\triangleleft}$ is a closure operator on $\mathcal{P}(W)$.
- N is nuclear iff γ_N is a nucleus.

Residuated frames cont.

Theorem [Galatos & Jipsen 2013]

$\mathbf{W}^+ := (\gamma_N[\mathcal{P}(W)], \cup_{\gamma_N}, \cap, \circ_{\gamma_N}, \backslash, /, \gamma_N(\{1\}))$,

$$X \cup_{\gamma_N} Y = \gamma_N(X \cup Y) \text{ and } X \circ_{\gamma_N} Y = \gamma_N(X \circ Y),$$

is a residuated lattice.

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Lemma [Galatos & Jipsen 2013]

All simple equations $[\mathrm{R}]$ are preserved by $(-)^+$:

$$\mathbf{W} \models (\mathrm{R}) \text{ iff } \mathbf{W}^+ \models [\mathrm{R}],$$

where

$$(\mathrm{R}): \quad (\forall r \in \mathrm{R})\, r(x_1, \ldots, x_n)\, N\, w \implies x_1 \circ \cdots \circ x_n\, N\, w$$

for all $x_1, \ldots, x_n \in W$ and $w \in W'$.

Basic equations as consequence relations

Let Var be a countable set of variables, and Var^* denote the free monoid generated by Var with identity 1.

For Γ a finite set of basic equations, we define $\vdash_\Gamma$ be the smallest relation on $\wp(\mathsf{Var}^*) \times \mathsf{Var}^*$ satisfying the following for all $X \subseteq \mathsf{Var}^*$

- $X \vdash_\Gamma x$ for all $x \in X$,
- For all $[A] = (a_0, A) \in \Gamma$, $u, v \in \mathsf{Var}^*$, and substitutions $\boldsymbol{\sigma}$,

$$(\forall a \in A)\ X \vdash_\Gamma \boldsymbol{\sigma}(uav) \implies X \vdash_\Gamma \boldsymbol{\sigma}(ua_0v).$$

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The relation $\vdash_\Gamma$ is a substitution invariant consequence relation.

Lemma

Let $\Gamma \cup \{[A]\}$ be a set of basic equations. Then

$$A \vdash_\Gamma a_0 \implies \mathsf{ISR} + \Gamma \models [A],$$

where $[A] = (a_0, A)$.

Proof.

Induct on the height of the proof-tree witnessing $A \vdash_\Gamma a_0$. □

Example

Let Φ_Γ be the associated closure operator on $\wp(\mathsf{Var}^*)$, i.e., $\Phi_\Gamma(X) = \{x : X \vdash_\Gamma x\}$.

Let $\Gamma = \{[\mathrm{R}]\}$ where $[\mathrm{R}] : x \leq x^2 \vee 1$. Let

$$[A] : \quad x^2 \quad \leq \quad x^5 \vee x$$

Here, $A := \{x^5, x\}$.

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Hence $A \vdash_\Gamma x^2$. In fact $\Phi_\Gamma(A) = \{x^5, x^4, x^3, x^2, x\}$.

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$xy \notin \Phi_\Gamma(B) = \{xy^2x, xyx, x^2\} \implies B \nvdash_\Gamma xy$

Does this imply $\mathsf{ISR} + \Gamma \not\models [B]$, or even $\mathsf{RL} + \Gamma \not\models [B]$?

Frame construction

For a finite set Σ of **simple equations**, define

- $W = \mathsf{Var}^*$
- $W' = \mathsf{Var}^* \times \wp(\mathsf{Var}^*) \times \mathsf{Var}^*$
- $N_\Sigma \subseteq W \times W'$ via

$$x \, N_\Sigma \, (u, X, v) \iff X \vdash_\Sigma uxv$$

Lemma

$\mathbf{W}_\Sigma = (W, W', N_\Sigma)$ is a residuated frame.

Proof.

$$xy \, N_\Sigma \, (u, X, v) \iff x \, N_\Sigma \, (u, X, yv) \iff y \, N_\Sigma \, (ux, X, v) \qquad \square$$

Equivalences

Theorem

Let Σ *be a set of simple equations. Then for a given* **proper** *equation* $[\mathrm{A}] = (a_0, \mathrm{A})$*, the following are equivalent:*

1. $\mathsf{RL} + \Sigma \models [\mathrm{A}]$.
2. $\mathrm{A} \vdash_{\Sigma} a_0$.
3. $\mathsf{ISR} + \Sigma \models [\mathrm{A}]$.

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Proof.

$(1 \Rightarrow 2)$: Since $[\mathrm{A}]$ is proper, it is ISR-equivalent to a simple equation $[\mathrm{R}] = (r_0, \mathrm{R})$ (i.e., r_0 is linear)

$$\begin{aligned} \mathrm{A} \nvdash_\Sigma a_0 \implies \mathrm{R} \nvdash_\Sigma r_0 \quad &\implies \quad r_0 \not\mathcal{N}_\Sigma (1, \mathrm{R}, 1) \\ &\implies \quad \mathbf{W}_\Sigma \not\models (\mathrm{R}) \\ &\implies \quad \mathbf{W}_\Sigma^+ \not\models [\mathrm{R}] \iff \mathbf{W}_\Sigma^+ \not\models [\mathrm{A}] \end{aligned}$$

But $\mathbf{W}_\Sigma^+ \in \mathsf{RL} + \Sigma$ since Σ is a set of simple equations. □

Non-proper equations

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By adding a bottom element $\perp$ to the signature and suitably defining a consequence relation $\vdash_{\Gamma\perp}$, we obtain a stronger correspondence.

Theorem

Let $\Gamma \cup \{[A]\}$ *be a set of basic equations where* $[A] = (a_0, A)$*. The following are equivalent:*

1. $\mathsf{RL} + \Sigma \models [A]$.
2. $A \vdash_{\Gamma\bot} a_0$.
3. $\mathsf{ISR}_\bot + \Sigma \models [A]$.

Let Γ be a set of basic equations.

Definition

- Γ is called *degenerate* if it contains a degenerate equation.
- For Γ not degenerate, the *simplification* of Γ is the set Σ_Γ containing all the equivalent simple equations from Γ.

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$\mathsf{RL} + \Gamma$ is the trivial variety if and only if Γ contains a degenerate equation.

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Corollary

If Γ is a non-degenerate set of basic equations, then

$$\mathsf{RL} + \Gamma \models [A] \iff A \vdash_\Sigma a_0,$$

where $[A] = (a_0, A)$ is an ISR-equation and $\Sigma = \Sigma_\Gamma$ is the simplification of Γ.

Definition

Let $[A]$ be an basic equation. We say $[A]$ is:

- **knotted** if $[A] : x^n \leq x^m$ for some $n \neq m$.
- **expansive** if $[A] : x^n \leq x^{n+c_1} \vee \cdots \vee x^{n+c_k}$, for some $n, k \geq 1$ and positive $c_1, \ldots, c_k$.
- **compressive** if $[A] : x^n \leq x^{n-c_1} \vee \cdots \vee x^{n-c_k}$, for some $n, k \geq 1$ and $1 \leq c_1, \ldots, c_k < n$.

○ For the above properties, we say $[A] = (a_0, A)$ is **pre-(property)** if there exists a substitution $\boldsymbol{\sigma}$ such that $[\boldsymbol{\sigma} A] = (\boldsymbol{\sigma} a_0, \boldsymbol{\sigma} A)$ is (property).

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○ For the above properties, we say $[A] = (a_0, A)$ is **pre-(property)** if there exists a substitution $\boldsymbol{\sigma}$ such that $[\boldsymbol{\sigma} A] = (\boldsymbol{\sigma} a_0, \boldsymbol{\sigma} A)$ is (property).

○ For a set of equations Σ, we say Σ is **pre-(property)** if it contains an equation that is pre-(property).

We say a variety $\mathcal{V} \subseteq \mathsf{RL}$ is *knotted*, *expansive*, or *compressive* if $\mathcal{V} \models [A]$ for some equation $[A]$ that is knotted, expansive, or compressive.

Theorem

Let Σ be a set of simple equations.

1. $\mathsf{RL} + \Sigma$ is knotted iff Σ is pre-knotted.
2. $\mathsf{RL} + \Sigma$ is expansive iff Σ is pre-expansive.
3. $\mathsf{RL} + \Sigma$ is compressive iff Σ is pre-compressive.
4. $\mathsf{RL} + \Sigma$ is integral iff Σ contains a strictly proper equation.

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Theorem

Let Σ be a set of simple equations.

1. $\mathsf{RL} + \Sigma$ is knotted iff Σ is pre-knotted.
2. $\mathsf{RL} + \Sigma$ is expansive iff Σ is pre-expansive.
3. $\mathsf{RL} + \Sigma$ is compressive iff Σ is pre-compressive.
4. $\mathsf{RL} + \Sigma$ is integral iff Σ contains a strictly proper equation.

A variety $\mathcal{V} \subseteq \mathsf{RL}$ is called **potent** if $\mathcal{V} \models x^n = x^m$ for some $n \neq m$.

Theorem

Let Σ be a set of simple equations. Then $\mathsf{RL} + \Sigma$ is potent if and only if Σ is pre-compressive [or resp. pre-expansive] and contains an expansive [resp. compressive] pre-knotted equation.

Decidability?

- For many sets Σ of simple equations, the equational theory for $\mathsf{ISR} + \Sigma$ is decidable.

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Open Question

Let $[\mathrm{R}]$ be any simple equation. Is the equational theory for $\mathsf{ISR} + [\mathrm{R}]$ decidable?

Thank you!