On the Number of Clonoids

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CU Boulder

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Mathematics

UNIVERSITY OF COLORADO **BOULDER**

Introduction

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- Post showed there are only countably many clones on a 2-element set. In contrast, there are continuum many such clonoids.

Minors and Clonoids

Let $[k] := \{1, \ldots, k\}$.

Definition

Let A, B be sets, $k \in \mathbb{N}$, and $f\colon A^k \to B$. For $\ell \in \mathbb{N}$ and $\sigma : [k] \to [\ell]$, the function

$$f^\sigma : A^\ell \to B, (x_1, \ldots, x_\ell) \mapsto f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

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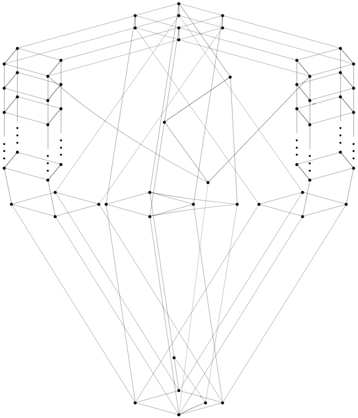
Let A be a set and $\mathbf{B} = (B, \mathcal{F})$ an algebra. A subset C of $\bigcup_{n\in\mathbb{N}} B^{A^n}$ is a *clonoid* with *source set* A and *target algebra* $\mathbf{B}$ if

1. C is closed under taking minors, and
2. for all $k \in \mathbb{N}$, the k-ary functions of C form a subalgebra of $\mathbf{B}^{A^k}$.

The set of all clonoids with source A and target algebra $\mathbf{B}$ is denoted $\mathcal{C}_{A,\mathbf{B}}$.

Post's Classification of Boolean Clones

Lattice of all clones on a two-element set $\{0, 1\}$, ordered by inclusion.



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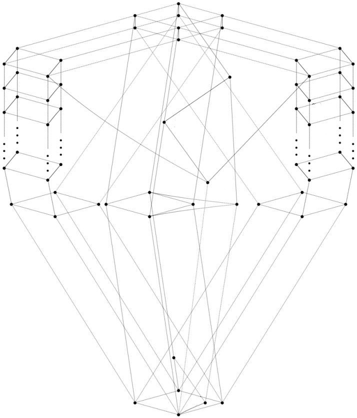


Question:

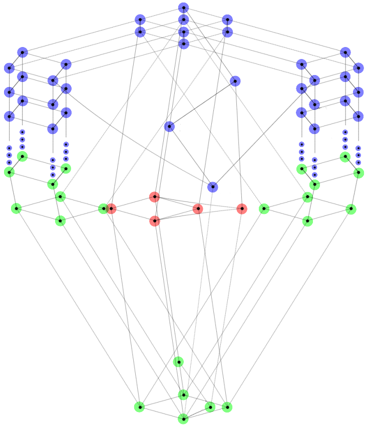
How many clonoids are there with a finite source A and a Boolean target algebra $\mathbf{B}$?

Number of Boolean Clonoids

Theorem (A.S., submitted 2018)

Let $\mathcal{C}_{A,\mathbf{B}}$ denote the set of all clonoids with finite source A ($|A| > 1$) and target algebra $\mathbf{B}$ of size 2. Then

1. $\mathcal{C}_{A,\mathbf{B}}$ is finite iff $\mathbf{B}$ has an near-unanimity (NU) term;
2. $\mathcal{C}_{A,\mathbf{B}}$ is countably infinite iff $\mathbf{B}$ has a cube term but no NU-term;
3. $\mathcal{C}_{A,\mathbf{B}}$ has size continuum iff $\mathbf{B}$ has no cube term.



Definition

A *k-cube term* of **B** is a $(2^k - 1)$-ary term c in the operations of **B** such that

$$c\underbrace{\begin{pmatrix} y & \dots \\ x & \dots \\ \vdots & \\ x & \dots \end{pmatrix}}_{\substack{\text{all } x,y\text{-columns} \\ \text{except } \overline{x}}} = \begin{pmatrix} x \\ x \\ \vdots \\ x \end{pmatrix}$$

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An *NU-term* of **B** is an k-ary $(k \geq 3)$ term f in the operations of **B** which satisfies

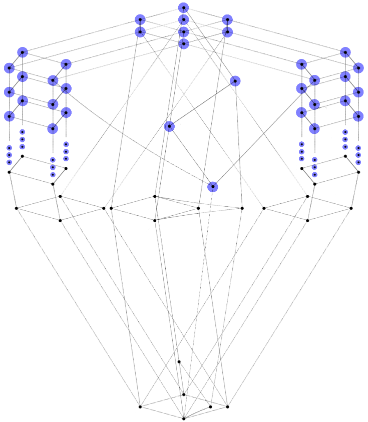
$$f(y, x, x, \dots, x, x) = f(x, y, x, \dots, x, x) = \dots = f(x, x, x, \dots, x, y) = x$$

for all $x, y \in B$.

Case 1: **B** has a NU-term

Proof Idea:

- Let $C \in \mathcal{C}_{A,\mathbf{B}}$ where **B** has a n-ary NU-term.



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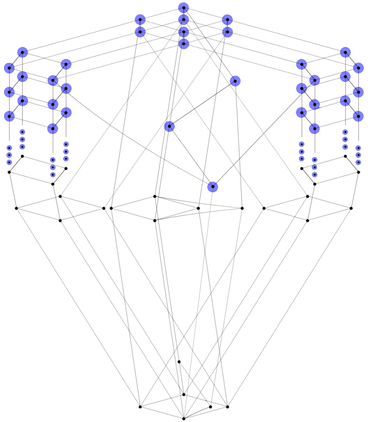

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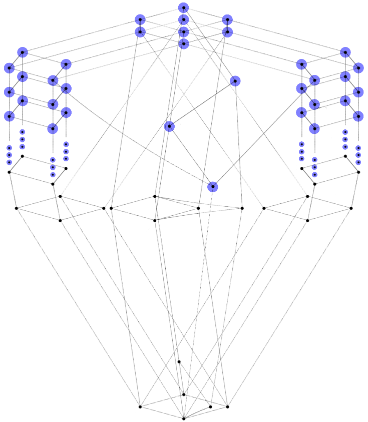

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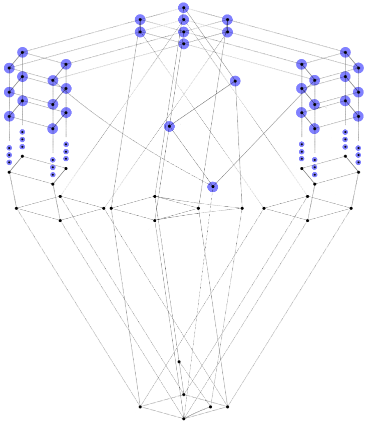
- $\mathcal{C}_{A,\mathbf{B}}$ is finite.



Case 2: **B** has a cube term but no NU-term

Note: If $\mathrm{Clo}(\mathbf{B}) \subseteq \mathrm{Clo}(\mathbf{B}')$, then

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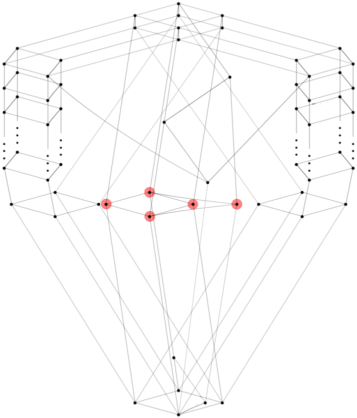

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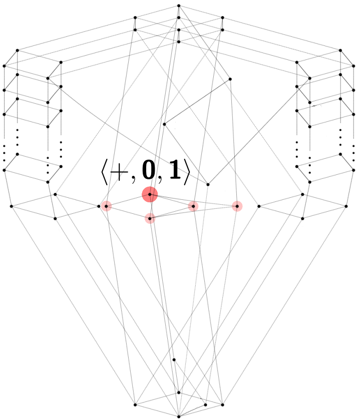
Enough to show

1. $|\mathcal{C}_{A,\mathbf{B}}| \leq \aleph_0$ when **B** has a cube term, and
2. $|\mathcal{C}_{A,\mathbf{B}}| = \aleph_0$ when $\mathrm{Clo}(\mathbf{B}) = \langle +, \mathbf{0}, \mathbf{1} \rangle$.



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Proof Idea (Aichinger, Mayr, 2016):
Show each $C \in \mathcal{C}_{A,\mathbf{B}}$ is finitely related, i.e. there exists a pair of relations (P, Q) such that C is the set of all functions that preserves (P, Q).

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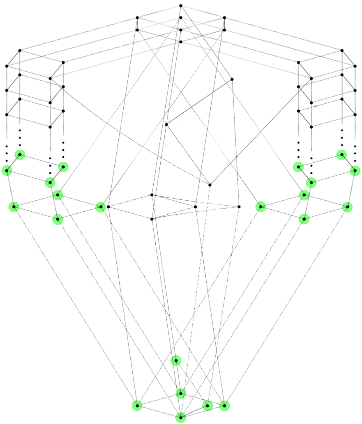
Proof Idea:
Construct an infinite family of clonoids with target algebra $\mathbf{B}$.
Let $0, 1 \in A$ and for $k \in \mathbb{N}$ define

$$e_k \colon A^k \to \{0,1\}, x \mapsto \begin{cases} 1 & \text{if } x = (1, \dots, 1), \\ 0 & \text{else.} \end{cases}$$

Show $\langle e_1 \rangle \subsetneq \langle e_2 \rangle \subsetneq \dots$

Case 3: **B** does not have a cube term



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Enough to show

$$|\mathcal{C}_{A,\mathbf{B}}| = 2^{\aleph_0}$$

when $\mathrm{Clo}(\mathbf{B})$ is one of the following:

- $\langle \wedge, \mathbf{0}, \mathbf{1} \rangle$
- $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$
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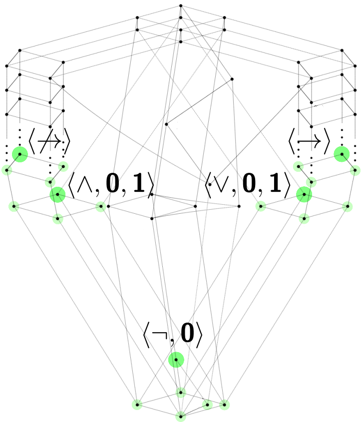

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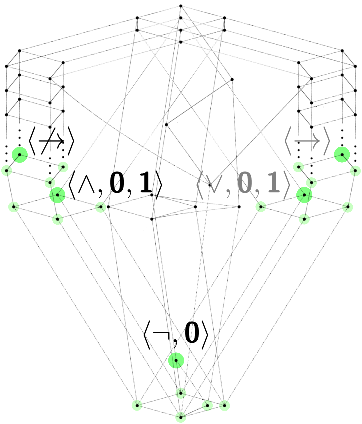

$\mathcal{C}_{A,\mathbf{B}}$ when $\mathbf{B} = (\{0,1\}, \wedge, \mathbf{0}, \mathbf{1})$

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Theorem (A.S., submitted 2018)

Let A be a finite set and $\mathbf{B}$ a finite idempotent algebra with $|A|, |B| > 1$. Then $\mathcal{C}_{A,\mathbf{B}}$ has size continuum iff $\mathbf{B}$ has no cube term.

Note

We have already discussed the forward direction.

B finite idempotent with no cube term

- Take a set A and finite idempotent algebra $\mathbf{B}$ without a cube term with $|A|, |B| > 1$.

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- Take a set A and finite idempotent algebra **B** without a cube term with $|A|, |B| > 1$.
- **B** must have cube term blocker (Kearnes, Szendrei, 2016), i.e. there exists a nonempty proper subset V of B such that

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- WLOG assume $0 \in V$ and $1 \in B \setminus V$. Thus

$$\{0,1\}^n \setminus \{(1,\ldots,1)\} \subseteq T_n \leq \mathbf{B}$$

Construction (cf. Yanov, Muchnik, 1959)

Let $P_n := \{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\} \subseteq A^n$.

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\begin{aligned}
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Claim: $\langle F_U \rangle_{\mathbf{B}} \cap F_{\mathbb{N}} = F_U$ for each $U \subseteq \mathbb{N}$.

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Lemma

f_k preserves (P_n, T_n) iff $k \neq n$.

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If $k = n$:

$$\begin{matrix} 1 & 0 & \cdots & 0 & \xrightarrow{f_k} & 1 \\ 0 & 1 & \cdots & 0 & \xrightarrow{f_k} & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \xrightarrow{f_k} & 1 \\ \pitchfork & \pitchfork & \cdots & \pitchfork & & \not\pitchfork \\ P_k & P_k & \cdots & P_k & & T_k. \end{matrix}$$

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If $k \neq n$:
For any $a_1, \ldots a_n \in P_n$,

$$f_k(a_1, \ldots, a_n)$$

has at least one zero entry. □

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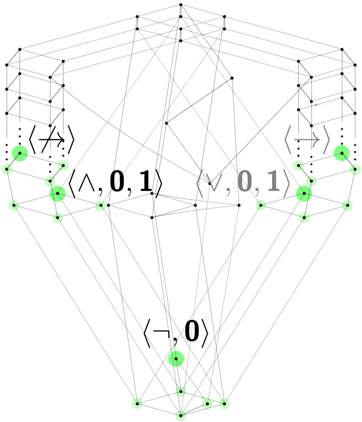
for $k_1, \dots, k_m \in U$, $n \in \mathbb{N} \setminus \{k_1, \dots, k_m\}$ and $\varphi \in \mathrm{Clo}(\mathbf{B})$.
Since all f_{k_i} preserve (P_n, T_n) and T_n is closed under φ, also

$$\varphi(f_{k_1}^{\sigma_1}, \dots, f_{k_m}^{\sigma_m}) \text{ preserves } (P_n, T_n).$$

However f_n does not preserves (P_n, T_n). □

Case 3: **B** (Boolean) does not have a cube term



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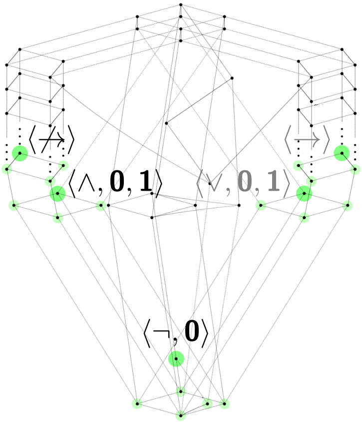

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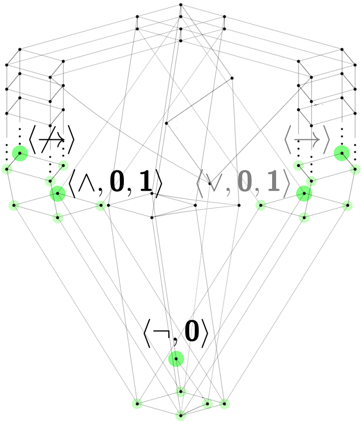

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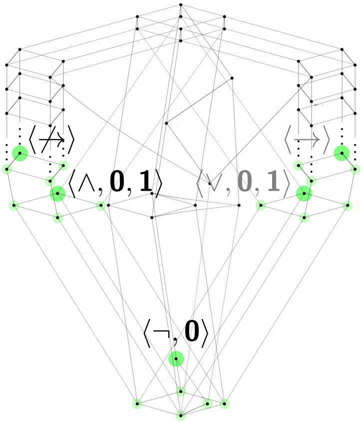

Corollary (A.S., submitted 2018)

For $m, n \geq 1$, there are continuum many clonoids from source $\{0, 1, \ldots, m\}$ into the target set $\{0, 1, \ldots, n\}$.

For **B** Boolean or idempotent $\mathcal{C}_{A,\mathbf{B}}$ is countable if **B** has a cube term; continuum otherwise.

Question: Does this generalize to clonoids with an arbitrary finite target algebra?