Algebras of fractions

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Introduction

Constructing algebras of fractions is a classical topic of abstract algebra: consider fields of quotients or Grothendieck groups.

In this talk, we review some classical results on groups of fractions, and show how more general algebras of fractions can be constructed.

This will yield some new and some known categorical equivalences between varieties of residuated structures (such as Heyting algebras).

Key novelty: groups of fractions can be constructed for (some) monoids, more general algebras of fractions can be constructed for **bimonoids**.

Groups of fractions

Can we embed a given monoid M into a group of fractions?

Definition. A **group of right fractions** of a monoid $\mathbf{M}$ is a group $\mathbf{G}$ such that $\mathbf{M}$ is (isomorphic to) a submonoid of $\mathbf{G}$ and moreover

$$\text{each } x \in \mathbf{G} \text{ has the form } x = a \cdot b^{-1} \text{ for some } a, b \in \mathbf{M}.$$

(Most of the results we discuss work also for semigroups with a few tweaks here and there. We only discuss monoids for the sake of simplicity.)

For a thorough account of the history of this problem see C. Hollings: *Mathematics Across the Iron Curtain: A History of the Algebraic Theory of Semigroups*.

Partially ordered groups of fractions

Can we embed a given pomonoid $\mathbf{M}$ into a pogroup of fractions?

Definition. A **pogroup of right fractions** of a pomonoid $\mathbf{M}$ is a pogroup $\mathbf{G}$ such that $\mathbf{M}$ is (isomorphic to) a subpomonoid of $\mathbf{G}$ and moreover

$$\text{each } x \in \mathbf{G} \text{ has the form } x = a \cdot b^{-1} \text{ for some } a, b \in \mathbf{M}.$$

Definition. A partially ordered monoid (group) or **pomonoid** (**pogroup**) is a monoid (group) equipped with a partial order with respect to which the multiplication is monotone: $a \leq b$ implies $a \cdot x \leq b \cdot x$ and $x \cdot a \leq x \cdot b$.

(We may also be interested in the lattice-ordered variant of the question. To avoid complications, we will mostly focus on the partially ordered problem.)

Overview of the first part of the talk

Two classical results ensure the existence of a group of fractions.

Existence of groups of fractions

- for commutative monoids (**Steinitz, 1910**)
- beyond the commutative case (**Ore, 1931**)

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For pomonoids with a division operation (so-called residuated posets), the existence theorems can be upgraded to a categorical equivalence.

Categorical equivalences

- certain residuated lattices and ℓ-groups with a conucleus (**Montagna & Tsinakis, 2010**)
- certain integral residuated lattices and ℓ-groups (**Bahls, Cole, Galatos, Jipsen & Tsinakis, 2003**)

Steinitz's Theorem

An early concern of abstract algebra was embedding a ring into a field of fractions, in the same way the ring $\mathbb{Z}$ embeds into the field $\mathbb{Q}$.

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From Steinitz's proof one can immediately extract:

Theorem. A commutative monoid embeds into an Abelian group of fractions if and only if it is cancellative.

Cancellative means: if $ax = bx$ or $xa = xb$, then $a = b$.

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The proof of the ordered version is almost identical:

Theorem. A commutative pomonoid embeds into an Abelian pogroup of fractions if and only if it is order-cancellative.

Order-cancellative means: if $ax \leq bx$ or $xa \leq xb$, then $a \leq b$.

Proof. Consider the set M^2, where $\langle a, b\rangle$ is to be interpreted as $a \cdot b^{-1}$.

Define the following relation on M^2: $\langle a, b\rangle \leqq \langle c, d\rangle \iff ad \leq bc$ in $\mathbf{M}$.

Informally: $ab^{-1} \leq cd^{-1} \iff ab^{-1}bd \leq cd^{-1}bd \iff ad \leq bc$.

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Define the following operation on M^2: $\langle a, b\rangle \cdot \langle c, d\rangle := \langle ac, bd\rangle$.

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The relation $\leqq$ is a preorder compatible with the above multiplication: the maps $\lambda_a : x \mapsto a \cdot x$ and $\rho_a : x \mapsto x \cdot a$ preserve the preorder.

(Transitivity relies on order-cancellativity.)

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The algebra of equivalence classes of $\leqq$ (ordered by $\leqq$) is a pogroup of fractions of $\mathbf{M}$. An element $a \in \mathbf{M}$ corresponds to the pair $\langle a, 1\rangle \in M^2$.

Ore's Theorem

Beyond the commutative case, things get a little tricky…

Theorem (Ore, 1931). A cancellative monoid $\mathbf{M}$ embeds into a group of right fractions if and only if it is **right reversible**:

$$\text{for each } a, b \in \mathbf{M} \text{ there are } x, y \in \mathbf{M} \text{ such that } ax = by.$$

Moreover, this group of right fractions $\mathbf{M}^{\div}$ is unique up to isomorphism. This theorem again extends to order-cancellative monoids.

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The question of which monoids embed into a group is rather delicate. Cancellativity itself is not sufficient but right reversibility is not necessary.

Fact (Maltsev, 1937). There is a monoid (semigroup) which embeds into some group but not into a group of fractions.

Necessary and sufficient conditions exist due to Maltsev, Pták, Lambek.

Proof. We only show uniqueness: if the group of fractions exists, it has to be given by the construction below. Checking existence is not too difficult.

We can transform left fractions into right fractions:

$$b^{-1}a = yx^{-1} \text{ whenever } ax = by \text{ (and such } x, y \text{ exist by reversibility).}$$

This tells us how to multiply right fractions: if $bx = cy$, then

$$ab^{-1} \cdot cd^{-1} = a(b^{-1}c)d^{-1} = a(xy^{-1})d^{-1} = ax(dy)^{-1}.$$

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It also tells us how to compare right fractions:

$$ab^{-1} \le cd^{-1} \iff ab^{-1}d \le c \iff ayx^{-1} \le c \iff ay \le cx$$

Now we again take the set M^2, interpret it as the algebra of formal fractions with the above multiplication, and factor by the preorder.

(There is also an elegant proof of Ore's Theorem using inverse semigroups.)

Residuated structures

This ends our review of classical results. Let us skip ahead some 70 years.

Partially ordered groups (pogroups) and lattice-ordered groups (ℓ-groups) now fit into a wider context of so-called residuated structures.

Definition. **Residuated posets** are pomonoids equipped with two division operations $a\backslash b$ and a/b:

$$b \leq a\backslash c \iff a \cdot b \leq c \iff a \leq c/b$$

Residuated lattices add a lattice order: $\langle A, \vee, \wedge, \cdot, 1, \backslash, / \rangle$.

Fact. In residuated lattices products distribute over joins:

$$a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c) \qquad \text{and} \qquad (a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c).$$

Examples of residuated structures

The residuation law is:

$$b \le a\backslash c \iff a \cdot b \le c \iff a \le c/b$$

For partially ordered groups $x\backslash y = x^{-1}y$ and $x/y = xy^{-1}$:

$$b \le a^{-1}c \iff a \cdot b \le c \iff a \le c \cdot b^{-1}$$

For Boolean algebras $x \cdot y = x \wedge y$ and $x\backslash y = \neg x \vee y$ and $x/y = x \vee \neg y$:

$$b \le \neg a \vee c \iff a \wedge b \le c \iff a \le c \vee \neg b$$

For Heyting algebras $x \cdot y = x \wedge y$ and $x\backslash y = x \to y$ and $x/y = y \to x$:

$$b \le a \to c \iff a \wedge b \le c \iff a \le b \to c$$

In the commutative case, we use the notation $x\backslash y = x \to y = y/x$.

Groups of fractions of residuated pomonoids

Can we say more about groups of fractions of residuated pomonoids?

Yes! Let $\mathbf{G}$ be the group of fractions of a residuated pomonoid $\mathbf{M}$. Then there is a well-defined map $\sigma : \mathbf{G} \to \mathbf{M}$ such that $\sigma(ab^{-1}) = a/b$.

Since $\mathbf{M}$ is a submonoid of $\mathbf{G}$, we can view σ as a map $\sigma : \mathbf{G} \to \mathbf{G}$.

This map $\sigma : \mathbf{G} \to \mathbf{G}$ is a **conucleus** on $\mathbf{G}$: interior operator such that $\sigma x \cdot \sigma y \leq \sigma(x \cdot y)$ and $\sigma(1) = 1$. Its image $\mathbf{G}_\sigma$ is isomorphic to $\mathbf{M}$.

Fact. The image of a conucleus σ on a residuated poset (lattice) can be equipped with the structure of a residuated poset (lattice).

Categorical equivalence for ℓ-groups with a conucleus

Theorem (Montagna & Tsinakis, 2010). The category of (Abelian) ℓ-groups $\mathbf{G}$ with a conucleus σ such that $\mathbf{G} \cong (\mathbf{G}_\sigma)^{\div}$ is equivalent to the category of cancellative right reversible (commutative) residuated lattices.

(An analogous theorem holds without assuming a lattice order.)

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(An analogous theorem holds without assuming a lattice order.)

Good: an existence theorem was upgraded to a categorical equivalence!

Bad: ℓ-groups with σ such that $\mathbf{G} \cong (\mathbf{G}_\sigma)^{\div}$ do not form a "natural" class.

We would prefer an equivalence between varieties or quasivarieties.

Negative cones of ℓ-groups

In an ℓ-group $\mathbf{G}$ the elements below 1 are called the **negative cone** of $\mathbf{G}$.

Fact. The negative cone of $\mathbf{G}$ forms an integral residuated lattice $\mathbf{G}^-$.

Integral means: 1 is the top element.

Fact. Each ℓ-group $\mathbf{G}$ is the group of fractions of $\mathbf{G}^-$: $\mathbf{G} \cong (\mathbf{G}^-)^{\div}$.

Proof. ℓ-groups satisfy $x = (1 \wedge x) \cdot (1 \vee x) = (1 \wedge x) \cdot (1 \wedge x^{-1})^{-1}$.

Fact. The projection onto the negative cone $x \mapsto 1 \wedge x$ is a conucleus.

Corollary. ℓ-groups form a full subcategory of ℓ-groups with a conucleus.

What do we get if we restrict the previous equivalence to this subcategory?

ℓ-groups are equivalent to their negative cones

Theorem (Bahls, Cole, Galatos, Jipsen, Tsinakis, 2003). The category of (Abelian) ℓ-groups is equivalent to the category of integral divisible cancellative (commutative) residuated lattices via $\mathbf{G} \mapsto \mathbf{G}^{-}$ and $\mathbf{M} \mapsto \mathbf{M}^{\div}$.

Divisible: $x \cdot (x \backslash (x \wedge y)) = x \wedge y = ((x \wedge y)/x) \cdot x$.

The definition of divisibility is not important now. What is important:

- the functor taking ℓ-groups to their negative cones is an equivalence
- the negative cones of ℓ-groups are definable by a set of equations
- an ℓ-group is determined up to isomorphism by its negative cone

Example: $\mathbb{Z}$ from $\mathbb{N}$

The naturals not only form a lattice-ordered monoid $\langle \mathbb{N}, \wedge, \vee, +, 0 \rangle$, they also come with a (dual) **residual**, namely the truncated subtraction:

$$a \dot{-} b := (a - b) \vee 0, \qquad\qquad a \dot{-} b \leq c \iff a \leq b + c.$$

Thanks to this structure, the equivalence class of each formal fraction $\langle a, b \rangle$ (interpreted as $a - b$) has a **canonical representative**: $\langle a, 0 \rangle$ or $\langle 0, a \rangle$.

These are defined equationally:

$$\langle a, b \rangle \text{ is canonical} \iff a \wedge b = 0.$$

Moreover, there is a projection map onto the set of canonical pairs:

$$\pi \langle a, b \rangle = \langle a \dot{-} b, b \dot{-} a \rangle.$$

Example: $\mathbb{Z}$ from $\mathbb{N}$

Instead of taking a quotient of $\mathbb{N}^2$ we can construct $\mathbb{N}^{\div}$ as follows:

- the universe of $\mathbb{N}^{\div}$ is the set of canonical pairs $\langle a, b\rangle$
- $\langle a, b\rangle + \langle c, d\rangle = \pi\langle a+c, b+d\rangle = \langle (a+c)\dot{-}(b+d), (b+d)\dot{-}(a+c)\rangle$
- the order is the restriction of the quotient order to canonical pairs:
  $$\langle a, b\rangle \leq \langle c, d\rangle \iff a+d \leq b+c$$

This works for the negative (or positive) cone of any Abelian ℓ-group!

The canonical pairs are precisely the pairs of the form $\langle 1 \wedge x, 1 \wedge x^{-1}\rangle$.

The hard part is therefore: to find an intrinsic description of such pairs (here: $a \wedge b = 0$) and to find the right projection terms (here: $\langle a\dot{-}b, b\dot{-}a\rangle$).

Review of the first part

Let us take stock of the results we have seen so far.

Steinitz, 1910:

Commutative $\mathbf{M}$: $\mathbf{M}^{\div}$ exists $\iff$ $\mathbf{M}$ is cancellative.

Ore, 1931:

Non-commutative $\mathbf{M}$: we rely on $a^{-1}b \mapsto xy^{-1}$.

Montagna & Tsinakis, 2010:

Residuated $\mathbf{M}$: cat. equivalence $\mathbf{M} \mapsto \langle \mathbf{M}^{\div}, \sigma \rangle$ and $\langle \mathbf{G}, \sigma \rangle \mapsto \mathbf{G}_{\sigma}$.

Bahls et al., 2003:

Special residuated $\mathbf{M}$: cat. equivalence $\mathbf{M} \mapsto \mathbf{M}^{\div}$ and $\mathbf{G} \mapsto \mathbf{G}^{-}$.

Reminder: examples of residuated structures

The residuation law is:

$$b \leq a \backslash c \iff a \cdot b \leq c \iff a \leq c / b$$

For partially ordered groups $x \backslash y = x^{-1} y$ and $x / y = x y^{-1}$:

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For Boolean algebras $x \cdot y = x \wedge y$ and $x \backslash y = \neg x \vee y$ and $x / y = x \vee \neg y$:

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$$b \leq a \to c \iff a \wedge b \leq c \iff a \leq b \to c$$

In the commutative case, we use the notation $x \backslash y = x \to y = y / x$.

Involutive residuated structures

As we have just seen, division sometimes splits into a complement-like or inverse-like operation $\overline{x}$ and a monoidal operation $x+y$:

$$a \backslash b = \overline{a} + b \qquad \text{and} \qquad a/b = a + \overline{b}.$$

For pogroups: $x+y=x\cdot y$. For Boolean algebras: $x+y=x\vee y$.

Other examples include MV-algebras and Sugihara monoids.

Such structures are called **involutive** residuated structures.

(The map $x \mapsto \overline{x}$ is an order-inverting involution.)

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Involutive residuated structures will serve as our algebras of fractions.

Involutive residuated structures

Fact. The two monoids are related by the De Morgan laws:

$$\overline{a+b} = \overline{b} \cdot \overline{a} \qquad \overline{a \cdot b} = \overline{b} + \overline{a} \qquad \overline{\overline{a}} = a$$

$$\overline{0} = 1 \qquad \overline{1} = 0$$

Examples: in pogroups $a \cdot b = (b^{-1} \cdot a^{-1})^{-1}$. In BAs $a \vee b = \neg(\neg a \wedge \neg b)$.

Fact. Complementation can be recovered from 0: $\overline{a} = a \backslash 0 = 0/a$.

Fact. Unital involutive residuated structures are simply unital residuated structures equipped with a certain "dualizing element" 0.

Bimonoids

The move from groups to involutive residuated pomonoids corresponds to the move from monoids to bimonoids.

Definition. A **bimonoid** is an ordered algebra $\langle A, \leq, \cdot, 1, +, 0\rangle$ such that $\langle A, \leq, \cdot, 1\rangle$ and $\langle A, \leq, +, 0\rangle$ are both pomonoids, and moreover

$$x \cdot (y + z) \leq (x \cdot y) + z \qquad \text{and} \qquad (x + y) \cdot z \leq x + (y \cdot z).$$

We call this condition **hemidistributivity** following Dunn & Hardegree.

An ℓ**-bimonoid** is a lattice-ordered bimonoid $\langle A, \vee, \wedge, \cdot, 1, +, 0\rangle$ such that

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z) \qquad x + (y \wedge z) = (x + y) \wedge (x + z)$$
$$(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z) \qquad (x \wedge y) + z = (x + z) \wedge (y + z)$$

In a **commutative** bimonoid both operations are commutative.

Complementation

Definition. Two elements x and y of a commutative bimonoid are called **complements** if

$$x \cdot y \leq 0 \qquad \text{and} \qquad 1 \leq y + x.$$

Fact. Each element x has at most one complement. We denote it $\overline{x}$.

Proof. If it had two complements y and y', then:

$$y \leq 1 \cdot y \leq (y' + x) \cdot y \leq y' + (x \cdot y) \leq y' + 0 \leq y'.$$

Fact. The map $x \mapsto \overline{x}$ is an order-inverting involution.

Fact. Bimonoids with complementation (i.e. with a unary map $x \mapsto \overline{x}$) are termwise equivalent to involutive residuated posets.

Examples of bimonoids

(Sub)reducts of involutive residuated posets: if $\langle A, \leq, \cdot, 1, +, 0, ^{-}\rangle$ is an involutive residuated poset, then $\langle A, \leq, \cdot, 1, +, 0\rangle$ is a bimonoid.

Pomonoids: take a pomonoid $\langle M, \leq, \cdot, 1\rangle$ and expand it by:

$$x + y = x \cdot y, \qquad 0 = 1, \qquad (\overline{x} = x^{-1}).$$

Distributive lattices: if $\langle L, \wedge, \vee, \top, \bot\rangle$ is a bounded distributive lattice, then the following defines an ℓ-bimonoid:

$$x \cdot y = x \wedge y, \qquad x + y = x \vee y, \qquad 1 = \top, \qquad 0 = \bot, \qquad (\overline{x} = \neg x).$$

Pointed Heyting algebras: if $\langle L, \wedge, \vee, \top, \bot, \to\rangle$ is a Heyting algebra and $0 \in L$, then the following defines an ℓ-bimonoid:

$$x \cdot y = x \wedge y, \qquad x + y = (0 \to (x \wedge y)) \wedge (x \vee y), \qquad 1 = \top.$$

Complemented envelopes

Let $\mathbf{B}$ be a commutative bimonoid in the following.

Definition. A **complemented envelope** of $\mathbf{B}$ is a complemented bimonoid $\mathbf{C}$ such that $\mathbf{B}$ is (isomorphic to) a sub-bimonoid of $\mathbf{C}$.

Definition. A **(commutative) bimonoid of fractions** of $\mathbf{B}$ is a commutative complemented envelope $\mathbf{F}$ such that

$$\text{each } x \in \mathbf{F} \text{ has the form } x = a + \overline{b} \text{ for some } a, b \in \mathbf{B},$$

or equivalently

$$\text{each } x \in \mathbf{F} \text{ has the form } x = a \cdot \overline{b} \text{ for some } a, b \in \mathbf{B}.$$

Fact. The commutative bimonoid of fractions $\mathbf{B}^{\div}$ is unique if it exists.

Complemented MacNeille completions

A commutative monoid embeds into an Abelian group if and only if it embeds into an Abelian group of fractions. Not so for bimonoids.

Stilll, each commutative bimonoid has a well-behaved envelope which is unique up to isomorphism: the **complemented MacNeille completion**.

Thm. Each commutative (ℓ-)bimonoid has a commutative complemented envelope where each x has the form $\bigwedge_{i \in I} a_i + \overline{b_i}$, or equivalently $\bigvee_{i \in I} a_i \cdot \overline{b_i}$.

(Each element is a meet of fractions, or equivalently a join of co-fractions.)

Example: for distributive lattices this is the MacNeille completion of the free Boolean extension (incidentally, this is precisely the injective hull).

Example: for cancellative commutative pomonoids this is the MacNeille completion of the pogroup of fractions. This is **not** a group: $\mathbb{N} \mapsto \mathbb{Z}_{-\infty}^{\infty}$.

Bimonoids of fractions

Let $\mathbf{B}^{\triangle}$ be the complemented MacNeille completion of a commutative $\mathbf{B}$.

Fact. $\mathbf{B}^{\div}$ exists $\iff$ elements of form $a + \overline{b}$ form a subalgebra of $\mathbf{B}^{\triangle}$,
$\iff$ elements of form $a \cdot \overline{b}$ form a subalgebra of $\mathbf{B}^{\triangle}$,
$\iff$ each fraction $a + \overline{b}$ is a co-fraction $x \cdot \overline{y}$,
$\iff$ each co-fraction $a \cdot \overline{b}$ is a fraction $x + \overline{y}$.

In that case $\mathbf{B}^{\div}$ is the algebra of fractions (equivalently, of co-fractions).

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How do we add and multiply fractions?

$$(a+\overline{b})+(c+\overline{d}) = a+c+\overline{b}+\overline{d} = (a+c)+\overline{b\cdot d}$$

$$(a+\overline{b})\cdot(c+\overline{d}) = x\overline{y}\cdot u\overline{v} = x\cdot u\cdot\overline{y}\cdot\overline{v} = xu\cdot\overline{y+v} = p+\overline{q}$$

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The above condition can be expressed by a (complicated) ∀∃∀-sentence in $\mathbf{B}$, which immediately yields the existence of $\mathbf{B}^{\div}$ in many cases.

This is an **existence theorem** for commutative bimonoids of fractions.

Bimonoids of fractions

Non-example: consider the chain $\mathsf{p} < \mathsf{q} < \mathsf{r} < \mathsf{s}$. The element $\mathsf{r} \cdot \overline{\mathsf{q}} = \mathsf{r} \wedge \overline{\mathsf{q}}$ does not have the form $a + \overline{b} = a \vee \overline{b}$ in any complemented envelope.

Bimonoids of fractions

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Bimonoids of fractions

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New example: a semilattice $x \cdot y = x \wedge y = x + y$ has a bimonoid of fractions if and only if for each a, b there is x such that $a \wedge x \leq b$ and for each p, q:

$$p \wedge x \leq b \ \& \ a \wedge q \leq b \implies p \wedge q \leq b$$

In particular, each Brouwerian semilattice has a bimonoid of fractions.

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New example: Boolean-pointed Brouwerian algebras, i.e. Brouwerian algebras with a constant 0 such that the interval $[0, \top]$ is Boolean.

(Previously only known for semilinear algebras.)

Brouwerian algebra: Heyting algebra without the assumption that $\bot$ exists.

Semilinear: subdirect product of chains, equivalently $x \to y \vee y \to x = \top$.

New categorical equivalence

We can upgrade the last existence theorem to an equivalence given again by the negative cone and bimonoid of fractions functors.

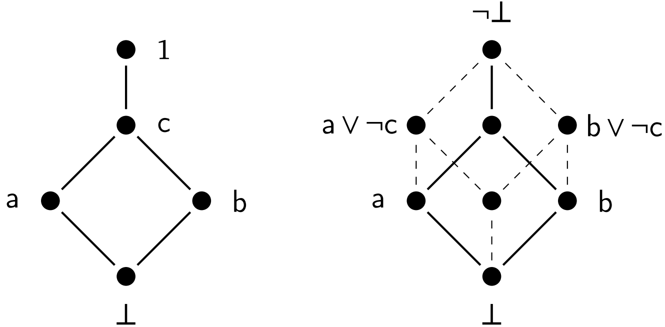
Theorem. The variety of Boolean-pointed Brouwerian algebras is categorically equivalent to the variety of commutative idempotent involutive residuated lattices with $x = (1 \wedge x) \cdot (0 \vee x)$.

Proof. Canonical pairs: $a \wedge b = 0$. Projection: $\langle (a \to ab) \to a0), a \to b \rangle$.

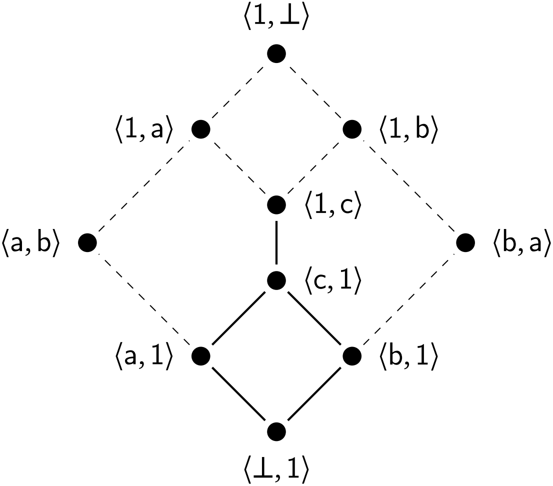
In fact, this projection term witnesses that a **single** cat. equivalence covers both of the extreme cases: Brouwerian algebras and the Abelian ℓ-groups.

We can now use our (good) knowledge of Brouwerian to add to our (poor) understanding of idempotent involutive RLs. **Example:** amalgamation.

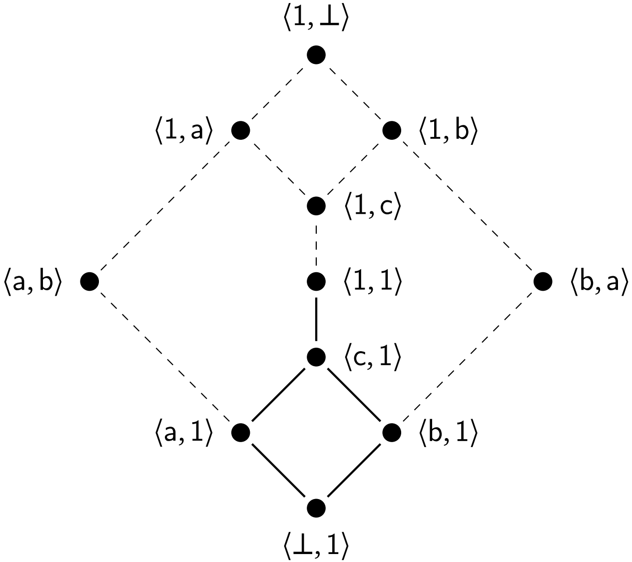
Example: the Boolean envelope of $\mathbf{H_5}$ $(0 = \bot)$



Example: the bimonoid of fractions of $\mathbf{H}_5$ $(0 = c)$



Example: the bimonoid of fractions of $\mathbf{H_5}$ $(0 = 1)$



Example: Sugihara monoids of fractions

Special case (Fussner & Galatos & Raftery 2012–17:

semilinear Brouwerian algebras $\sim$ odd Sugihara monoids.

Semilinear Boolean-pointed Brouwerian algebras $\sim$ Sugihara monoids.

Definition. An **(odd) Sugihara monoid** is a distributive idempotent commutative involutive residuated lattice (such that $0 = 1$).

(These form an algebraic semantics for the relevance logic R-Mingle.)

Example: $\langle \mathbb{Z}, \wedge, \vee, \odot, 0, \oplus, 0, - \rangle$ where

$$x \odot y = \begin{cases} x & \text{if } |x| > |y|, \\ y & \text{if } |y| > |x|, \\ x \wedge y & \text{if } |x| = |y|, \end{cases} \qquad x \oplus y = \begin{cases} x & \text{if } |x| > |y|, \\ y & \text{if } |y| > |x|, \\ x \vee y & \text{if } |x| = |y|. \end{cases}$$

This algebra is precisely the bimonoid of fractions of the negative integers viewed as a Brouwerian algebra ($a \to b = 0$ if $a \leq b$, $a \to b = b$ else).

Non-commutative bimonoids of fractions

For non-commutative bimonoids of fractions, we only have a weak analogue of Ore's Theorem: a (useful) sufficient condition.

Theorem. A given bimonoid has a bimonoid of fractions if (i) we can transform fractions into co-fractions and vice versa, and (ii) there are four functions satisfying a certain system of quasiequations telling us how to solve $\overline{a} + b = x + \overline{y}$ and $a \cdot \overline{b} = \overline{x} \cdot y$.

This allows us to handle the ℓ-group case and extend it to the case of so-called ℓ-pregroups satisfying $1 \wedge (x \vee y) = (1 \wedge x) \vee (1 \wedge y)$.

In the latter case we have a quasiequational description of the negative cones and a categorical equivalence as in the ℓ-group case.

Non-commutative bimonoids of fractions

$$\alpha_\circ(x, y \cdot z) = \alpha_\circ(x, y) \cdot \alpha_\circ(\beta_\circ(x, y), z)$$
$$\beta_\circ(x + y, z) = \beta_\circ(x, z) + \beta_\circ(y, \alpha_\circ(x, z))$$
$$\alpha_\circ(x + y, z) = \alpha_\circ(y, \alpha_\circ(x, z))$$
$$\beta_\circ(x, y \cdot z) = \beta_\circ(\beta_\circ(x, y), z)$$

$$\alpha_+(x, y \cdot z) = \alpha_+(\beta_+(x, z), y) \cdot \alpha_+(x, z)$$
$$\beta_+(x + y, z) = \beta_+(x, \alpha_+(y, z)) + \beta_+(y, z)$$
$$\alpha_+(x + y, z) = \alpha_+(x, \alpha_+(y, z))$$
$$\beta_+(x, y \cdot z) = \beta_+(\beta_+(x, z), y)$$

$$\alpha_\circ(0, x) = x \qquad \beta_\circ(x, 1) = x \qquad \alpha_+(0, x) = x \qquad \beta_+(x, 1) = x$$
$$\alpha_\circ(x, 1) = 1 \qquad \beta_\circ(0, x) = 0 \qquad \alpha_+(x, 1) = 1 \qquad \beta_+(0, x) = 0$$

$$x \cdot \alpha_\circ(y, z) \leq w + \beta_\circ(y, z) \iff \alpha_+(y^{rr}, x) \cdot z \leq \beta_+(y^{rr}, x) + w$$
$$x \cdot \alpha_\circ(y, z) \leq w + \beta_\circ(y, z) \iff \alpha_+(w, z^{\ell\ell}) \cdot x \leq \beta_+(w, z^{\ell\ell}) + y$$

Conclusion

Classical existence theorems for groups of fractions can be upgraded to categorical equivalences in the presence of residuation.

More general algebras of fractions can be constructed:

(certain) pomonoids $\mapsto$ pogroups of fractions

(certain) bimonoids $\mapsto$ involutive residuated posets of fractions

We know when commutative algebras of fractions exist. We know that the commutative complemented MacNeille completion always exists.

Open problem: describe the bimonoidal subreduct of MV-algebras (recent related work by Cabrer, jipsen & Kroupa).

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Thank you for your attention!