Profinite completions of MV-algebras

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Definition of an MV-algebra

An MV-algebra can be defined as an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying:

MV-1: $(A, \oplus, 0)$ is an Abelian monoid;

MV-2: $\neg : A \to A$ is an involution;

MV-3: $1 := \neg 0$ is absorbant;

MV-4: $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Note: The rule $x \leq y$ iff $\neg x \oplus y = 1$, defines a bounded distributive lattice order on A.

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3. The following subalgebras of $[0, 1]$. For every $n \geq 2$, $Ł_n = [0, 1] \cap \mathbb{Z}\frac{1}{n-1} = \left\{ \frac{k}{n-1} : 0 \leq k \leq n-1 \right\}$.

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4. Let X be any topological space and $A := \mathcal{C}(X, [0, 1])$ be the set of continuous functions from $X \to [0, 1]$.
Given $f, g \in A$ and $x \in X$,
   - $(\neg f)(x) = 1 - f(x)$
   - $(f \oplus g)(x) = Min(1, f(x) + g(x))$

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4. **Ex.** Let X be any topological space and $A := \mathcal{C}(X, [0,1])$ be the MV-algebra above. For every $p \in X$, let

   $$O_p := \{f \in A : f \text{ vanishes on some neighborhood } U \text{ of } p\}$$

   Then O_p is an ideal of A.

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5. As in most varieties of algebras, the is a natural isomorphism $\widehat{A} \cong \left\{\alpha \in \prod_{I \in \mathrm{id}_f(A)} A/I : \phi_{JI}(\alpha(I)) = \alpha(J) \text{ whenever } I \subseteq J\right\}$

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- **We add to this literature the profinite completion of MV-algebras**

Main Theorem

For every MV-algebra A, its profinite completion is algebraically and topologically isomorphic to

$$\prod_{M \in \mathrm{Max}_f(A)} A/M$$

Where $\mathrm{Max}_f(A)$ denotes the set of maximal ideals of A with finite rank, i .e., $A/M \cong Ł_n$ for some integer $n \geq 2$.

Sketch of the proof-1

The proof will use the following important lemmas.

- **Lemma 1:** If $I \in \mathrm{id}_f(A)$, then there exist $M_1, M_2, \dots, M_r \in \mathrm{Max}_f(A)$ such that $I = M_1 \cap M_2 \cap \dots \cap M_r$ and $A/I \cong \prod_{i=1}^{r} A/M_i$.

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- **Lemma 2:** The ideal I is uniquely determined by the set $\mathcal{S}(I) := \{M_1, M_2, \ldots, M_r\} \subseteq \mathrm{Max}_f(A)$.

- **Lemma 3:** For every $I, J \in \mathrm{id}_f(A)$, if $I \subseteq J$, then $\mathcal{S}(J) \subseteq \mathcal{S}(I)$.

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- Consider the embedding

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- Using the lemma and other tools, one proves that the homomorphism thus obtained is a a topological and algebraic isomorphism.

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$$\widehat{B} \cong \mathcal{P}(X)$$
- An MV-algebra A is isomorphic to its profinite completion if and only if A is profinite and every maximal ideal of A of finite rank is principal.
- Let A be a regular MV-algebra in which every maximal ideal has finite rank. Then,
$$B(\widehat{A}) \cong \widehat{B(A)}$$

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- **Sketch of proof:** Given an MV-algebras homomorphism $\varphi : A \to B$, define $\widehat{\varphi} : \widehat{A} \to \widehat{B}$ by $\widehat{\varphi}(\alpha)(N) := \alpha(\varphi^{-1}(N))$ for $\alpha \in \prod_{M \in \mathrm{Max}_f(A)} A/M$ and $N \in \mathrm{Max}_f(B)$

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- **Remark:** The profinite completion is a functor when considered over any variety of algebras and the corresponding Stone algebras. In our case, we simply get a more concrete description.

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- In particular, the only one with infinite rank comes from the point at infinity, while the other maximal ideals have the $Ł_{n+1}$'s for quotients.
- It follows from the Main thoerem that $\widehat{B} \cong \prod_{n=1}^{\infty} Ł_{n+1}$

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- This leads to a natural isomorphism Θ from Cont(Z) onto Cont(Y), namely $\Theta(f) = f \circ \overline{\tau}$. Take $A' = \Theta(B)$, which is easily verified to be a separating subalgebra of Cont(Y) and satisfy (i)&(ii).

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- This leads to a natural isomorphism Θ from $\mathrm{Cont}(Z)$ onto $\mathrm{Cont}(Y)$, namely $\Theta(f) = f \circ \overline{\tau}$. Take $A' = \Theta(B)$, which is easily verified to be a separating subalgebra of $\mathrm{Cont}(Y)$ and satisfy (i)&(ii).
- Finally, the density clause can be added by simply replacing Y with the closure of X if needed.

Classes of profinite MV-algebras that are profinite completion of some MV-algebras

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- For every $t \in [0, 1]$ and $\epsilon > 0$, $(t - \epsilon, t + \epsilon) \cap Ł_{n_x} \neq \emptyset$, for all but finitely many $x \in X$.

Sketch of the proof of 2. above

We may assume that X is infinite.

- Topologize X so that the space obtained is the one-point compactification of the discrete space $X \setminus \{x_0\}$

Remark: Just like the case of Heyting algebras, it remains unclear whether or not there exists a profinite MV-algebra that is not the profinite completion of some MV-algebras.

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- And $C/J_{x_0} \cong [0,1]$
- Thus $A \cong \widehat{C}$.

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THANK YOU FOR YOUR ATTENTION !!!!!!!!

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