Finite Degree Clones Are Undecidable

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1. Clones and The Finite Degree Problem
2. The Encoding of Computation
3. Non-halting Implies Infinite Degree
4. Halting Implies Finite Degree
5. Conclusion and Open Problems

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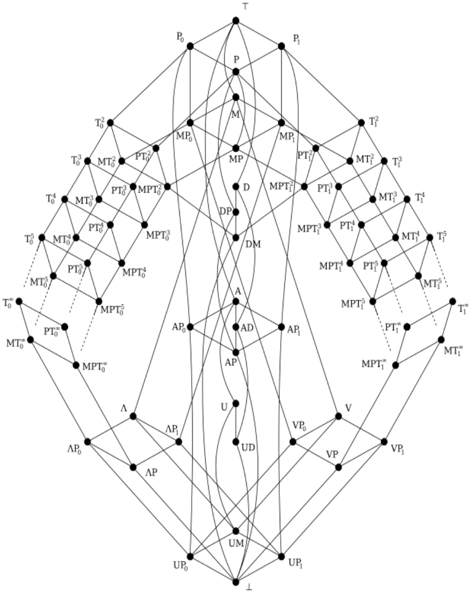
A **clone** is a set of finitary operations closed under

- composition,
- variable identification,
- variable permutation,
- introduction of extraneous variables.

Emil Post in 1941 famously classified all Boolean clones.

Over ($\geq$ 3)-element domains structure is quite complicated.



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Clones are infinite. How can they be an input to an algorithm?

A clone on finite domain A can be **finitely specified** in essentially 2 ways.

First way: Given $\mathcal{F}$, a finite set of operations of A, define $\mathrm{Clo}(\mathcal{F}) =$ "the smallest clone containing $\mathcal{F}$".

- A with $\mathcal{F}$ forms a algebra, $\mathbb{A} = \langle A; \mathcal{F} \rangle$. Define $\mathrm{Clo}(\mathbb{A}) = \mathrm{Clo}(\mathcal{F})$.
- A **relation** of $\mathbb{A}$ is a subpower $R \subseteq A^n$ closed under $\mathcal{F}$ (hence $\mathrm{Clo}(\mathcal{F})$)
- Define $\mathrm{Rel}_n(\mathbb{A}) = \mathrm{Rel}_n(\mathcal{F}) =$ "all $(\leq n)$-ary relations of $\mathbb{A}$".
- Define $\mathrm{Rel}(\mathbb{A}) = \mathrm{Rel}_n(\mathcal{F}) = \bigcup_{n<\infty} \mathrm{Rel}_n(\mathbb{A})$

These are the **finitely generated** clones.

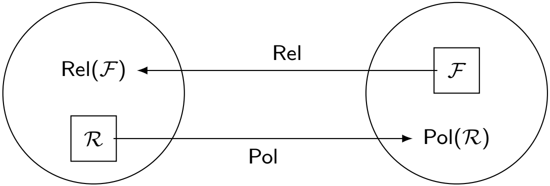
Second way: Given $\mathcal{R}$, a finite set of subpowers of A, define $\mathrm{Pol}(\mathcal{R}) =$ "the set of all operations of A preserving all subpowers in $\mathcal{R}$".

These are the **finitely related**/**finite degree** clones.

$$\mathrm{Rel}(\mathcal{F}) = \{R \subseteq A^n \mid R \text{ is preserved by all operations in } \mathcal{F}\}$$
$$\mathrm{Pol}(\mathcal{R}) = \{f : A^n \to A \mid f \text{ preserves all subpowers in } \mathcal{R}\}$$

These two operators form a **Galois connection**.

$$\mathcal{R} \subseteq \mathrm{Rel}(\mathcal{F}) \iff \mathcal{F} \subseteq \mathrm{Pol}(\mathcal{R})$$



Every Galois connection defines two closure operators. Here, they are

$$\mathrm{Clo} = \mathrm{Pol} \circ \mathrm{Rel} \qquad \text{and} \qquad \mathrm{RClo} = \mathrm{Rel} \circ \mathrm{Pol}\,.$$

If $\mathbb{R} \in \mathrm{RClo}(\mathcal{S})$, then we say "$\mathcal{S}$ entails $\mathbb{R}$" and write $\mathcal{S} \models \mathbb{R}$.

If $f \in \mathrm{Pol}(\mathcal{S})$, then we say "$\mathcal{S}$ entails f" and write $\mathcal{S} \models f$.

For a set of relations $\mathcal{S}$, define

$$\deg(\mathcal{S}) = \sup \left\{ \text{arity}(\mathbb{R}) \mid \mathbb{R} \in \mathcal{S} \right\}.$$

For a clone $\mathcal{C}$, define

$$\deg(\mathcal{C}) = \inf \left\{ \deg(\mathcal{S}) \mid \text{Pol}(\mathcal{S}) = \mathcal{C} \right\}.$$

For an algebra $\mathbb{A}$, define

$$\deg(\mathbb{A}) = \deg(\text{Clo}(\mathbb{A})).$$

The Finite Degree Problem

`Input:` finite algebra $\mathbb{A} = \langle A; f_1, \ldots, f_n \rangle$ generating clone $\mathcal{C}$

`Output:` whether $\deg(\mathcal{C}) < \infty$

(seems to originate in the 70s with the study of lattices of clones over domains of more than 2 elements)

The Finite Degree Problem

`Input:` finite algebra $\mathbb{A} = \langle A; f_1, \ldots, f_n \rangle$ generating clone $\mathcal{C}$
`Output:` whether $\deg(\mathcal{C}) < \infty$

Given a Minsky machine $\mathcal{M}$, we encode it into a finite algebra $\mathbb{A}(\mathcal{M})$.

Theorem

The following are equivalent.

- $\mathcal{M}$ *halts,*
- $\deg(\mathbb{A}(\mathcal{M})) < \infty$ *(i.e.* $\mathbb{A}(\mathcal{M})$ *is finitely related),*

Similar approaches have proved the following are undecidable:

- finite residual bound (McKenzie)
- finite axiomatizability/Tarski's problem (McKenzie)
- certain omitting types (McKenzie, Wood)
- existence of a term op. that is NU on all but 2 elements (Maroti)
- DPSC, leading to another solution to Tarski's problem (M)
- profiniteness (Nurakunov and Stronkowski)

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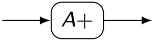
A Minsky machine has

- registers A and B that have integer values ≥ 0,
- instructions to add 1 to a register,



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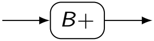
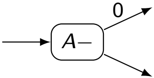


- instructions to test if a register is 0 and otherwise subtract 1 from it.



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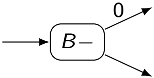
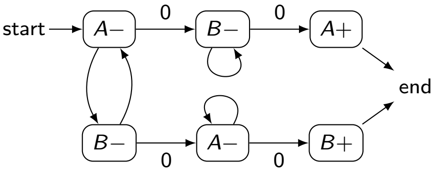
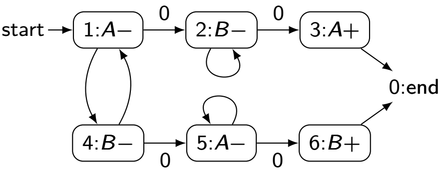


We can represent a Minsky machine as a finite flow graph.

$\mathcal{M} =$





Step	State	A	B
0	(1,	2,	3)
1	(4,	1,	3)
2	(1,	1,	2)
3	(4,	0,	2)
4	(1,	0,	1)
5	(2,	0,	1)
6	(2,	0,	0)
7	(3,	0,	0)
8	(0,	1,	0)

How to represent intermediate computations?

- Assign a **state** to each node.
- A **configuration** (i, α, β) represents each stage of computation.
- Consider $\mathcal{M}$ as a function, and write

$$\mathcal{M}(i, \alpha, \beta) = (j, \alpha', \beta') \qquad \text{or} \qquad \mathcal{M}^n(i, \alpha, \beta) = (j, \alpha', \beta')$$

  (single step of computation or multiple).
- On (α, β), $\mathcal{M}$ halts with registers $(1, 0)$ if $\alpha \leq \beta$ and $(0, 1)$ otherwise.

The encoding of computation

- let $\mathbb{A}(\mathcal{M})$ be the algebra we intend to build
- configurations (i, α, β) $\quad\leftrightsquigarrow\quad$ special elements of $A(\mathcal{M})^n$
- term operations should simulate the action of $\mathcal{M}$ (need placemarker, $\bullet$)
- computation on configurations $\quad\leftrightsquigarrow\quad$ subalgebra generation

$\mathbb{A}(\mathcal{M})$ has universe... $\quad A(\mathcal{M}) = \left\{ \langle i, c \rangle \mid i \text{ a state of } \mathcal{M},\ c \in \{A, B, 0, \bullet, \times\} \right\}$

Given configuration (k, α, β) and $n \in \mathbb{N}$ define a subset of $\mathbb{A}(\mathcal{M})^n$,

$$\mathtt{conf}(k, \alpha, \beta) = \bigcup_{p \in P_n} \Big\{ p\Big(\underbrace{\langle k, A \rangle, \ldots, \langle k, A \rangle}_{\alpha}, \underbrace{\langle k, B \rangle, \ldots, \langle k, B \rangle}_{\beta}, \underbrace{\langle k, 0 \rangle, \ldots, \langle k, 0 \rangle}_{n-\alpha-\beta-1}, \langle k, \bullet \rangle \Big) \Big\}$$

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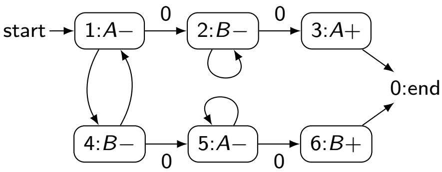
The encoding of computation

- term operations should simulate the action of $\mathcal{M}$
- computation on configurations $\rightsquigarrow$ subalgebra generation



Term operations

- $M(x, y)$ for $\boxed{R+}$ or $\boxed{R-}$
- $M'(x)$ for $\boxed{R-} \xrightarrow{0}$

Design considerations

- $M(r, s) = t$ if and only if...
  - $r, s \in \mathtt{conf}(i, \alpha, \beta)$
  - $r \neq s$
  - $t \in \mathtt{conf}(\mathcal{M}(i, \alpha, \beta))$ via some $\boxed{R+}$ or $\boxed{R-}$

- $M'(r) = t$ if and only if...
  - $r \in \mathtt{conf}(i, \alpha, \beta)$
  - $t \in \mathtt{conf}(\mathcal{M}(i, \alpha, \beta))$ via some $\boxed{R-} \xrightarrow{0}$

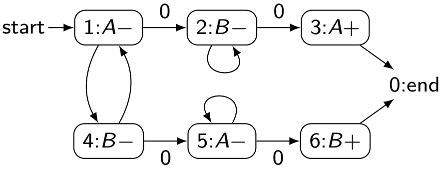
- otherwise introduce $\times$ into the output t

Can we actually define M and M' with these features?

$$M(x,y) = \begin{cases} \langle j, R \rangle & \text{if } x = \langle i, \bullet \rangle,\ y = \langle i, 0 \rangle,\ \boxed{i : R+} \longrightarrow \boxed{j : *}, \\ \langle j, 0 \rangle & \text{if } x = \langle i, \bullet \rangle,\ y = \langle i, R \rangle,\ \boxed{i : R-} \longrightarrow \boxed{j : *}, \\ \langle j, \bullet \rangle & \text{if } x = \langle i, 0 \rangle,\ y = \langle i, \bullet \rangle,\ \boxed{i : R+} \longrightarrow \boxed{j : *}, \\ \langle j, \bullet \rangle & \text{if } x = \langle i, R \rangle,\ y = \langle i, \bullet \rangle,\ \boxed{i : R-} \longrightarrow \boxed{j : *}, \\ \langle j, c \rangle & \text{if } x = y = \langle i, c \rangle,\ \boxed{i : R+} \longrightarrow \boxed{j : *} \text{ or } \boxed{i : R-} \longrightarrow \boxed{j : *}, \\ \langle j, \times \rangle & \text{else if } x = \langle i, c \rangle,\ y = \langle i, d \rangle,\ \boxed{i : R+} \longrightarrow \boxed{j : *} \text{ or } \boxed{i : R-} \longrightarrow \boxed{j : *}, \\ \langle i, \times \rangle & \text{otherwise, where } y = \langle i, c \rangle. \end{cases}$$

$$M'(x) = \begin{cases} \langle k, c \rangle & \text{if } x = \langle i, c \rangle,\ \boxed{i : R+} \xrightarrow{0} \boxed{k : *},\ c \neq R, \\ \langle k, \times \rangle & \text{else if } x = \langle i, R \rangle,\ \boxed{i : R+} \xrightarrow{0} \boxed{k : *}, \\ \langle i, \times \rangle & \text{otherwise, where } x = \langle i, c \rangle. \end{cases}$$

Let's see an example computation...



Step	State	A	B
0	1	2	1
1	4	1	1
2	1	1	0
3	4	0	0
4	5	0	0
5	6	0	0
6	0	0	1

$$\textbf{1:}\quad M\begin{pmatrix} \langle 1,\bullet\rangle, \langle 1,A\rangle \\ \langle 1,A\rangle, \langle 1,\bullet\rangle \\ \langle 1,A\rangle, \langle 1,A\rangle \\ \langle 1,B\rangle, \langle 1,B\rangle \end{pmatrix} = \begin{pmatrix} \langle 4,0\rangle \\ \langle 4,\bullet\rangle \\ \langle 4,A\rangle \\ \langle 4,B\rangle \end{pmatrix} \qquad \textbf{4:}\quad M'\begin{pmatrix} \langle 4,0\rangle \\ \langle 4,0\rangle \\ \langle 4,\bullet\rangle \\ \langle 4,0\rangle \end{pmatrix} = \begin{pmatrix} \langle 5,0\rangle \\ \langle 5,0\rangle \\ \langle 5,\bullet\rangle \\ \langle 5,0\rangle \end{pmatrix}$$

$$\textbf{2:}\quad M\begin{pmatrix} \langle 4,0\rangle, \langle 4,0\rangle \\ \langle 4,\bullet\rangle, \langle 4,B\rangle \\ \langle 4,A\rangle, \langle 4,A\rangle \\ \langle 4,B\rangle, \langle 4,\bullet\rangle \end{pmatrix} = \begin{pmatrix} \langle 1,0\rangle \\ \langle 1,0\rangle \\ \langle 1,A\rangle \\ \langle 1,\bullet\rangle \end{pmatrix} \qquad \textbf{5:}\quad M'\begin{pmatrix} \langle 5,0\rangle \\ \langle 5,0\rangle \\ \langle 5,\bullet\rangle \\ \langle 5,0\rangle \end{pmatrix} = \begin{pmatrix} \langle 6,0\rangle \\ \langle 6,0\rangle \\ \langle 6,\bullet\rangle \\ \langle 6,0\rangle \end{pmatrix}$$

$$\textbf{3:}\quad M\begin{pmatrix} \langle 1,0\rangle, \langle 1,0\rangle \\ \langle 1,0\rangle, \langle 1,0\rangle \\ \langle 1,A\rangle, \langle 1,\bullet\rangle \\ \langle 1,\bullet\rangle, \langle 1,A\rangle \end{pmatrix} = \begin{pmatrix} \langle 4,0\rangle \\ \langle 4,0\rangle \\ \langle 4,\bullet\rangle \\ \langle 4,0\rangle \end{pmatrix} \qquad \textbf{6:}\quad M\begin{pmatrix} \langle 6,0\rangle, \langle 6,0\rangle \\ \langle 6,0\rangle, \langle 6,\bullet\rangle \\ \langle 6,\bullet\rangle, \langle 6,0\rangle \\ \langle 6,0\rangle, \langle 6,0\rangle \end{pmatrix} = \begin{pmatrix} \langle 0,0\rangle \\ \langle 0,\bullet\rangle \\ \langle 0,B\rangle \\ \langle 0,0\rangle \end{pmatrix}.$$

Takeaways on a relation $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^n$ ...

- certain elements of R encode configurations of $\mathcal{M}$,
- M and M' encode the action of $\mathcal{M}$ in the presence of these elements.

$$\texttt{conf}(k, \alpha, \beta) = \bigcup_{p \in P_n} \left\{ p\Big(\underbrace{\langle k, A \rangle, \dots, \langle k, A \rangle}_{\alpha}, \underbrace{\langle k, B \rangle, \dots, \langle k, B \rangle}_{\beta}, \underbrace{\langle k, 0 \rangle, \dots, \langle k, 0 \rangle}_{n-\alpha-\beta-1}, \langle k, \bullet \rangle \Big) \right\}$$

Questions

- What if R doesn't contain these kinds of elements?
- What if R contains elements that aren't "computational"? (multiple $\bullet$'s or non-constant states)

Call $\mathbb{R}$ **computational** if it doesn't contain any elements with 2 $\bullet$'s or non-constant state.

The **capacity** of a computation $\mathcal{M}^k(i, \alpha, \beta) = (j, \alpha', \beta')$ is the max sum of the registers.

The **capacity** of computational $\mathbb{R}$ is (number of coordinates with $\bullet$)-1.

We consider the halting problem on **0 register input**: config $= (1,0,0)$.

Let $\mathbb{S}_m = \mathrm{Sg}_{\mathbb{A}(\mathcal{M})^m}\big(\mathtt{conf}(1,0,0)\big)$.

Theorem (The Coding Theorem)

- *If* $\mathcal{M}^n(1,0,0) = (k,\alpha,\beta)$ *has capacity* $< m$ *then* $\mathtt{conf}(k,\alpha,\beta) \subseteq S_m$.
- *If* $\mathtt{conf}(k,\alpha,\beta) \subseteq S_m$ *and* $\mathcal{M}$ *does not halt with capacity* $< m$ *then* $\mathcal{M}^n(1,0,0) = (k,\alpha,\beta)$ *for some* n *and has capacity* $< m$.

Corollary

The following are equivalent.

- $\mathcal{M}$ *halts with capacity* $< m$,
- $\mathbb{S}_m$ *is halting (i.e. contains* $\mathtt{conf}(0,\alpha,\beta)$*),*
- *every computational* $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^\ell$ *with capacity* $\geq m$ *is halting.*

Theorem (The Coding Theorem)

- *If* $\mathcal{M}^n(1,0,0) = (k,\alpha,\beta)$ *has capacity* $< m$ *then* $\texttt{conf}(k,\alpha,\beta) \subseteq S_m$.
- *If* $\texttt{conf}(k,\alpha,\beta) \subseteq S_m$ *and* $\mathcal{M}$ *does not halt with capacity* $< m$ *then* $\mathcal{M}^n(1,0,0) = (k,\alpha,\beta)$ *for some* n *and has capacity* $< m$.

Framework for proving the hardness of algebraic properties

- Start out with $\mathbb{A}(\mathcal{M}) = \langle A(\mathcal{M}) \,;\; M, M' \rangle$.
- Add operations so that the property is recognizable in $\mathrm{Rel}(\mathbb{A}(\mathcal{M}))$ $\big($ ideally in the $(\mathbb{S}_m)_{m\in\mathbb{N}}$ $\big)$.
- Use a computer to verify necessary computations.
- Use software development techniques:
  write unit tests, rapidly iterate the operation definitions.

This allows us to give a more unified construction for the previously mentioned undecidability results in Universal Algebra.

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Observe

$$\deg(\mathcal{C}) = \infty \quad \text{if and only if} \quad \forall n\ \ \mathrm{Rel}_n(\mathcal{C}) \not\models \mathrm{Rel}(\mathcal{C})$$
$$\text{if and only if} \quad \forall n\ \exists \mathbb{R}\ \ \mathrm{Rel}_n(\mathcal{C}) \not\models \mathbb{R}$$

Idea: to show that $\deg(\mathbb{A}(\mathcal{M})) = \infty$ when $\mathcal{M}$ does not halt, we show the last equivalence holds for $\mathcal{C} = \mathrm{Clo}(\mathbb{A}(\mathcal{M}))$.

Two operations involved

- semilattice operation $\wedge$
  locally flat: $a \wedge b \neq \langle *, \times \rangle$ iff $a = b$
- "initialization" operation $I(x, y)$
  returns any configuration to $\mathtt{conf}(1, 0, 0)$

At this point $\mathbb{A}(\mathcal{M}) = \langle A(\mathcal{M})\ ;\ M, M', \wedge, I \rangle$.

$\mathrm{Rel}_n(\mathcal{C}) \models \mathbb{R}$ if and only if $\mathbb{R}$ can be built from $\mathrm{Rel}_n(\mathcal{C})$ using

- intersection of equal arity relations,
- (cartesian) product of finitely many relations,
- permutation of the coordinates of a relation, and
- projection of a relation onto a subset of coordinates.

Theorem (Zadori 1995)

$\mathrm{Rel}_n(\mathbb{A}) \models \mathbb{S}$ *if and only if*

$$\mathbb{S} = \pi\left(\bigcap_{i\in I} \mu_i\Big(\prod_{j\in J_i} \mathbb{R}_{ij}\Big)\right)$$

for some $\mathbb{R}_{ij} \in \mathrm{Rel}_n(\mathbb{A})$, *some coordinate projection* π, *and some coordinate permutations* μ_i.

Lemma

Suppose that

$$\mathtt{conf}(1,0,0) \subseteq \pi\left(\bigcap_{i\in I} \mu_i\left(\prod_{j\in J_i} \mathbb{R}_{ij}\right)\right) = \mathbb{S} \le \mathbb{A}(\mathcal{M})^m,$$

where π is a projection, the μ_i are permutations, and the $\mathbb{R}_{ij}$ are a finite collection of members of $\mathrm{Rel}_n(\mathbb{A}(\mathcal{M}))$, and $n < m$. Then $\mathbb{S}$ is halting.

Theorem

The following hold for any Minsky machine $\mathcal{M}$.

- *If $\mathcal{M}$ does not halt with capacity m then $m < \deg(\mathbb{A}(\mathcal{M}))$.*
- *If $\mathcal{M}$ does not halt then $\mathbb{A}(\mathcal{M})$ is not finitely related.*

Proof: Suppose that $\deg(\mathbb{A}(\mathcal{M})) \le m$. This implies in particular that $\mathrm{Rel}_m(\mathbb{A}(\mathcal{M})) \models \mathbb{S}_{m+1}$. By Zadori's theorem, $\mathbb{S}_{m+1}$ can be represented as in the Lemma above, so by that same Lemma it is halting. By the Coding Theorem, this implies that $\mathcal{M}$ halts with capacity m, a contradiction. ·

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Strategy

- The relations $\mathbb{S}_m$ witnessed non-entailment when $\mathcal{M}$ did not halt. When $\mathcal{M}$ does halt, these relations eventually witness the halting.
- Show that for some suitably chosen k, we have $\mathrm{Rel}_k(\mathbb{A}(\mathcal{M})) \models \mathrm{Rel}_n(\mathbb{A}(\mathcal{M}))$ for all n.
- We proceed by induction on n.
- The base case of $n = k$ is trivial.
- We thus endeavor to prove $\mathrm{Rel}_{n-1}(\mathbb{A}(\mathcal{M})) \models \mathbb{R}$ for $\mathbb{R} \in \mathrm{Rel}_n(\mathbb{A}(\mathcal{M}))$.
- Relations in $\mathrm{Rel}_n(\mathbb{A}(\mathcal{M}))$ can be divided into 4 different kinds, so we proceed by cases.
- We add operations to handle entailment in each of the different cases: $N_\bullet(w, x, y, z)$, $P(u, v, x, y)$, $H(x, y)$, $N_0(x, y, z)$, $S(x, y, z)$.
- $\mathbb{A}(\mathcal{M}) = \langle A(\mathcal{M})\ ;\ M, M', \wedge, I, N_\bullet, P, H, N_0, S \rangle$ (final version)

$$\mathbb{A}(\mathcal{M}) = \langle A(\mathcal{M})\ ;\ M, M', \wedge, I, N_\bullet, P, H, N_0, S \rangle$$

Case $\mathbb{R}$ is non-computational

- There is an element with 2 •'s or with non-constant state.
- 2 •'s: operation $N_\bullet$ handles entailment.
- Non-constant state: operation P handles entailment.

Theorem

If $m \geq 3$ and $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^m$ is non-computational then $\mathrm{Rel}_{m-1}(\mathbb{A}(\mathcal{M})) \models \mathbb{R}$.

Case $\mathbb{R}$ is halting

- R contains an element of $\mathtt{conf}(0,0,0)$.
- Any element of $\mathtt{conf}(0,0,0)$ can be used with operations I, H, and N_0 to prove entailment.

Theorem

If $3 \leq m$ and $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^m$ is halting then $\mathrm{Rel}_{m-1}(\mathbb{A}(\mathcal{M})) \models \mathbb{R}$.

We are left to examine computational non-halting $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^n$.

Let's say that $\mathcal{M}$ halts with capacity κ.

Two metrics (both subsets of $[n]$)

- $\mathcal{D}(\mathbb{R}) =$ "coordinates i such that $\exists r \in R$ with $r(i) = \langle j, \bullet \rangle$"
  $=$ "the $\bullet$ (dot) part of $\mathbb{R}$."
- $\mathcal{N}(\mathbb{R}) =$ "the inherently non-halting part of $\mathbb{R}$" ...
  - $\pi_{\mathcal{N}(\mathbb{R})}(\mathbb{R})$ is non-halting,
  - if $K = \left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right|$ then $\mathbb{S}_K \leq \mathbb{R}$.

Case $\mathbb{R}$ is computational and $|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})| > \kappa$

- $\left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right| > \kappa$ then $\mathbb{R}$ contains a halting subalgebra.
- it follows that $\mathbb{R}$ halts!

We thus consider computational non-halting $\mathbb{R}$ with $\left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right| \leq \kappa$.

Case computational non-halting $\mathbb{R}$ with $\left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right| \leq \kappa$

Theorem

Assume that $n \geq \kappa + 16$ and

- $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^n$ *is computational non-halting,*
- $\left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right| \leq \kappa$,
- $\vdots$ *(several technical hypotheses)*

Then $\mathrm{Rel}_{n-1}(\mathbb{A}(\mathcal{M})) \models \mathbb{R}$.

This completes the case analysis!

Theorem

If $\mathcal{M}$ halts with capacity κ then $\deg(\mathbb{A}(\mathcal{M})) \leq \kappa + 16$.

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Theorem

The following are equivalent.

- $\mathcal{M}$ *halts,*
- $\deg(\mathbb{A}(\mathcal{M})) < \infty$ *(i.e.* $\mathbb{A}(\mathcal{M})$ *is finitely related),*
- $\mathcal{M}$ *halts with capacity at least* $\deg(\mathbb{A}(\mathcal{M})) - 16$*.*

Interesting observations

- There are infinitely many $\mathcal{M}$ with halting status independent of ZFC.
- Thus, there are infinitely many $\mathcal{M}$ such that $\deg(\mathbb{A}(\mathcal{M})) < \infty$ is independent of ZFC.
- There are finite algebras $\mathbb{A}$ that whose finite-relatedness is independent of ZFC.
- $$\text{maxdeg}_\sigma(n) = \sup\left\{ \deg(\mathbb{A}) \;\middle|\; \begin{array}{c} \mathbb{A} \text{ has signature } \sigma, \\ \deg(\mathbb{A}) < \infty, \text{ and } |A| \leq n \end{array} \right\}$$

  is not computable.

Finite Generation Problems

Problem

Given relations $\mathcal{R}$, decide if $\mathcal{C} = \mathrm{Pol}(\mathcal{R})$ is finitely generated.
That is, decide whether $\mathcal{C} = \mathrm{Clo}(\mathcal{F})$ for some finite set of operations $\mathcal{F}$.

Problem

Given relations $\mathcal{R}$ and operations $\mathcal{F}$, decide whether $\mathrm{Pol}(\mathcal{R}) = \mathrm{Clo}(\mathcal{F})$.

Natural Duality Problems

We can modify the definition of $\deg(\cdot)$ to obtain a duality degree: $\deg_\partial(\cdot)$.

Problem (Finite Duality Degree)

Decide whether $\deg_\partial(\mathbb{A}) < \infty$ for finite $\mathbb{A}$.

Duality entailment implies usual entailment, so we already have that $\mathbb{A}(\mathcal{M})$ is not finitely duality related when $\mathcal{M}$ does not halt.

Problem

If $\mathcal{M}$ halts, is $\deg_\partial(\mathbb{A}(\mathcal{M})) < \infty$?

Problem

Given finite $\mathbb{A}$, decide whether $\mathbb{A}$ admits a duality. .

Finite Degree Clones Are Undecidable

Theorem

The following are equivalent.

- $\mathcal{M}$ *halts,*
- $\deg(\mathbb{A}(\mathcal{M})) < \infty$ *(i.e.* $\mathbb{A}(\mathcal{M})$ *is finitely related),*
- $\mathcal{M}$ *halts with capacity at least* $\deg(\mathbb{A}(\mathcal{M})) - 16$.

Thank you for your attention.