Infinite games for teams

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Defining team games

Suppose Γ is a game where *I* and *II* take turns playing sequences of fixed length τ.

I	$p_0^0, \ldots, p_{\tau-1}^0$		$p_0^1, \ldots, p_{\tau-1}^1$		$\cdots$
II		$q_0^0, \ldots, q_{\tau-1}^0$		$q_0^1, \ldots, q_{\tau-1}^1$	$\cdots$

- Call *I* a **team** of size τ.
- Call $(p_i^0, p_i^1, p_i^2, \ldots)$ the plays of **player** I_i.
- Call a strategy σ for team *II* **independent** if each q_i^n played according to σ depends only on p_i^m for $m < n$.
- In other words, each player II_i following an independent strategy for *II* ignores I_j and II_j for $j \neq i$.
- (Use analogous terminology for *I*.)

A minimal example

In general, a team may have a winning strategy but not an independent winning strategy. For example:

- I plays two bits a, b and then II plays two bits c, d.
- II wins iff $d = a$.
- Team II can always win, but player II_1, playing d, needs to know what I_0 played (a).

Semi-independent strategies

$$\begin{array}{llllll} I & p^0_0, \ldots, p^0_{\tau-1} & & p^1_0, \ldots, p^1_{\tau-1} & & \cdots \\ II & & q^0_0, \ldots, q^0_{\tau-1} & & q^1_0, \ldots, q^1_{\tau-1} & \cdots \end{array}$$

- Call a strategy σ for team II **semi-independent** if each q^n_i played according to σ depends only on p^m_j for $m < n$ and $j \leq i$.
- In other words, each player II_i following a semi-independent strategy for II ignores I_j and II_j for $j > i$.
- (Use analogous terminology for I.)

Minimal example revisited

Same minimal example as before:

- I plays two bits a, b and then II plays two bits c, d.
- II wins iff $d = a$.
- Team II has a semi-independent winning strategy:

$$c = 0 \text{ and } d = a.$$

Modifying the example:

- II wins iff $c = b$.
- Now team II has a winning strategy but not a semi-independent winning strategy: II_0, playing c, needs to know what I_1 played (b).

A better modification:

- II wins iff $c = a \wedge b$ and $d = a \vee b$.
- Now II has a winning strategy, but each player of team II needs to know what both players on team I played.

The product Banach-Mazur game for teams

- The game $\mathrm{BM}^{\Pi}_{\tau}(A, X)$:
  - Let $X = \prod_{i<\tau} X_i$ be a nonempty topological product space, $A \subset X$, and $1 \leq \tau < \omega$.
  - For each $i < \tau$, I_i and II_i play open subsets of X_i

$$\begin{array}{lccccc} I & U_i^0 & & U_i^1 & & \cdots \\ II & & V_i^0 & & V_i^1 & \cdots \end{array}$$

    such that $U_i^0 \supset V_i^0 \supset U_i^1 \supset V_i^1 \supset \cdots$.
  - II wins iff $\prod_{i<\tau} \bigcap_{n<\omega} V_i^n \subset A$.

- II has a winning strategy iff II has a semi-independent winning strategy iff A is comeager.

- II has an independent winning strategy iff A contains a product of τ comeager sets.

The group Banach-Mazur game for teams

- The game $\mathrm{BM}^{\mathrm{group}}_{\tau}(A, G)$:
  - Let $(G, \cdot)$ be topological group, $A \subset G$, and $1 \leq \tau < \omega$.
  - For each $i < \tau$, I_i and II_i play open subsets of G

I	U_i^0		U_i^1		$\cdots$
II		V_i^0		V_i^1	$\cdots$

    such that $U_i^0 \supset V_i^0 \supset U_i^1 \supset V_i^1 \supset \cdots$.
  - II wins iff $x_0 \cdot x_1 \cdots x_{\tau-1} \in A$ for all $x \in \prod_{i<\tau} \bigcap_{n<\omega} V_i^n$.

- If $\tau \geq 2$, then II has a winning strategy iff II has an independent winning strategy iff $A = G$.

Proof idea: If $g \in G$ and $Y, Z \subset G$ are comeager, then $g = y \cdot z$ for some $(y, z) \in Y \times Z$.

A product measure game for teams

The game $\mathcal{N}^{\Pi}_{\tau}(A, X, \varepsilon)$:

- Let $1 \leq \tau < \omega$ and $0 < \varepsilon \in \mathbb{R}$.
- For each $i < \tau$, let μ_i be a regular Borel measure on a topological space X_i.
- Let $X = \prod_{i<\tau} X_i$ be the product space and let $\mu = \prod_{i<\tau} \mu_i$ be the product measure.
- Let $A \subset X$.
- For each round $n < \omega$, for each $i < \tau$, I_i and II_i play finite sequences of open subsets of X_i

I	$\cdots$	$U^n_{i,0}, \ldots, U^n_{i,a_n}$		$\cdots$
II	$\cdots$		$V^n_{i,0}, \ldots, V^n_{i,b_n}$	$\cdots$

  such that the sequence lengths a_n, b_n are independent of i.
- $U = \bigcup_{n<\omega} \bigcup_{j<a_n} \prod_{i<\tau} U_{i,j}$ and $V = \bigcap_{n<\omega} \bigcup_{j<b_n} \prod_{i<\tau} V_{i,j}$.
- II wins iff $A \supset V \not\subset U$ or $\mu(U) > \varepsilon$.

Strategies for the measure game

• If A has outer measure less than ε and is Lindelöf, then I has an independent winning strategy.

- I covers A by open boxes with total measure $\leq \varepsilon$.
- I completely ignores II's plays.

• If A has inner measure greater than ε, then II has a semi-independent winning strategy.

- Each sequence $V^n_{i,0}, \ldots, V^n_{i,b_n}$ played includes lots of repetition if $i < \tau - 1$ and includes lots of instances of $\varnothing$ if $i > 0$.

The club game for teams

- $[S]^\omega$ is the set of countably infinite subsets of S.
- $\mathcal{C} \subset [S]^\omega$ is *club* iff $\mathcal{C}$ is closed with respect to union of increasing ω-chains and every $X \in [S]^\omega$ is contained in some $Y \in \mathcal{C}$.

- The club game $\text{Club}_\tau(S, \mathcal{E})$ for team size $\tau < \omega_1$:
  - Let S be an uncountable set S and $\mathcal{E} \subset [S]^\omega$.
  - *I* and *II* play τ-sequences of elements of S for ω rounds.

$$\begin{array}{lccccc} I & (p_i^0)_{i<\tau} & & (p_i^1)_{i<\tau} & & \cdots \\ II & & (q_i^0)_{i<\tau} & & (q_i^1)_{i<\tau} & \cdots \end{array}$$

  - *II* wins iff $\bigcup_{i<\tau}\{p_i^0, q_i^0, p_i^1, q_i^1, p_i^2, q_i^2, \ldots\} \in \mathcal{E}$.

- *II* (*I*) has a winning strategy iff *II* (*I*) has a semi-independent winning strategy iff $\mathcal{E}$ contains (avoids) a club.

- *II* (*I*) has an independent winning strategy iff there is a club $\mathcal{C} \subset [S]^\omega$ such that $\bigcup_{i<\tau} X_i \in \mathcal{E}$ ($\notin \mathcal{E}$) for all $X_0, \ldots, X_{\tau-1} \in \mathcal{C}$.

An elementary submodel proof

Claim: *II* has a independent winning strategy for $\mathrm{Club}_\tau(S, \mathcal{E})$ if and only if there is a club $\mathcal{C} \subset [S]^\omega$ such that $\bigcup_{i<\tau} X_i \in \mathcal{E}$ for all $X_0, \ldots, X_{\tau-1} \in \mathcal{C}$.

Proof of "only if":

- Suppose σ is an independent winning strategy for *II*.
- Suppose for each $i < \tau$ that $(M_i, \in)$ is a countable elementary substructure of a sufficiently large fragment of the universe $(V, \in)$.
- Suppose $\sigma \in M_i$ for each $i < \tau$.
- The set $\mathcal{C}$ of all possible $M_0 \cap S$ is a club.
- Let each player I_i enumerate $M_i \cap S$.
- Since $\sigma \in M_i$ and is independent, player II_i, following σ, will play only in $M_i \cap S$.
- Since σ is winning, $\bigcup_{i<\tau}(M_i \cap S) \in \mathcal{E}$.

$\aleph_1$ vs. $\aleph_2$

• If $|S| = \aleph_1$ and $\mathcal{E} \subset [S]^\omega$, then *II* has a winning strategy for $\text{Club}_1(S, \mathcal{E})$ iff *II* has an independent winning stategy for $\text{Club}_\tau(S, \mathcal{E})$.

• Proof: Every club subset of ω_1 contains a club that is also a chain.

• If $|S| \geq \aleph_2$ and $1 \leq \tau < \omega$, then there is $\mathcal{E}$ such that *II* has an independent winning strategy for $\text{Club}_\tau(S, \mathcal{E})$ but not for $\text{Club}_{\tau+1}(S, \mathcal{E})$.

• Proof outline:

- Assume $\omega_2 \subset S$.
- Let $\mathcal{E}$ be the set of all $\bigcup_{i<\tau}(N_i \cap S)$ where each N_i is a countable elementary submodel.
- Given a club $\mathcal{C}$, there are $\tau + 1$ countable elementary submodels $M_0, \ldots, M_\tau$ such that $M_i \cap S \in \mathcal{C}$ and:

$$\begin{array}{ccccc} \sup(M_0 \cap \omega_1) & > & \cdots & > & \sup(M_\tau \cap \omega_1) \\ \sup(M_0 \cap \omega_2) & < & \cdots & < & \sup(M_\tau \cap \omega_2) \end{array}$$

The relative completeness game for teams

- Natural examples of clubs come from finitary closure properties.
- Example: the set of all countable subalgebras of a fixed algebra.
- But the union of two subalgebras need not be a subalgebra.

Definition

The *relative completeness game* $\mathrm{RC}_\tau(A)$:

- *I* and *II* play τ-sequences of elements of A for ω rounds.

I	$(p_i^0)_{i<\tau}$		$(p_i^1)_{i<\tau}$		$\cdots$
II		$(q_i^0)_{i<\tau}$		$(q_i^1)_{i<\tau}$	$\cdots$

- *II* wins iff $\bigcup_{i<\tau}\{p_i^0, q_i^0, p_i^1, q_i^1, p_i^2, q_i^2, \ldots\}$ **generates** a relatively complete subalgebra of A.
- A subalgebra B of a Boolean algebra A is *relatively complete* if every principal ideal of A restricts to one of B:

$$\forall a \in A\ \exists b \in B \quad B \cap {\downarrow} a = {\downarrow} b.$$

A game characterization of projective Boolean algebras

Definition

A Boolean algebra A is *projective* if it a retract of some free Boolean algebra F. (Retract means $A \underset{r}{\leftarrow} F \underset{e}{\leftarrow} A$; $r \circ e = \mathrm{id}$)

(The topological dual of this concept a retract of a power of 2, *a.k.a.*, a *Dugundji* space.)

Theorem

If A is a Boolean algebra, then the following are equivalent.

- *A is projective.*
- *For each $\tau < \omega$, II has an independent winning strategy for $RC_\tau(A)$.*
- *II has an independent winning strategy for $RC_\omega(A)$.*
- *For each ordinal τ, II has an independent winning strategy for $RC_\tau(A)$.*

$\aleph_n$ vs. $\aleph_{n+1}$

Theorem
If $1 \leq n < \omega$, A is a Boolean algebra, and $|A| \leq \aleph_n$, then the following are equivalent.

- *A is projective.*
- *II has an independent winning strategy for $RC_n(A)$.*
- *For each ordinal $\tau \geq n$, II has an independent winning strategy for $RC_\tau(A)$.*

Theorem
If $1 \leq n < \omega$, then there is a Boolean algebra of size $\aleph_{n+1}$ such that II has an independent winning strategy for $RC_n(A)$ but not for $RC_{n+1}(A)$.

These last three Boolean algebra theorems, translated into claims about clubs, are proved in arXiv:1607.07944.

Stationary strategies

- A strategy σ for II in a game is *stationary* if each nth play of II depends only on the nth play of I.

- The *fast* relative completeness game $\mathrm{RC}_\tau^{\mathrm{fast}}(A)$:
  - I plays τ-sequences of elements of A and II plays τ-sequences of finite **subsets** of A for ω rounds.

I	$(p_i^0)_{i<\tau}$		$(p_i^1)_{i<\tau}$		$\cdots$
II		$(B_i^0)_{i<\tau}$		$(B_i^1)_{i<\tau}$	$\cdots$

  $p_i^n \in B_i^n$ is required of II.
  - II wins iff $\bigcup_{i<\tau} \bigcup_{n<\omega} B_i^n$ generates a relatively complete subalgebra of A **or** some $\bigcup_{n<\omega} B_i^n$ is not a subalgebra of A.

- The *closed* relative completeness game $\mathrm{RC}_\tau^{\mathrm{closed}}(A)$:
  - Like $\mathrm{RC}_\tau^{\mathrm{fast}}(A)$, but now each B_i^n must also be a **subalgebra**.

Stationary independent strategies

The following are equivalent.

- *II* has an independent winning strategy for $\mathsf{RC}_\tau(A)$.
- *II* has an independent winning strategy for $\mathsf{RC}_\tau^{\mathsf{fast}}(A)$.
- *II* has an independent winning strategy for $\mathsf{RC}_\tau^{\mathsf{closed}}(A)$.
- *II* has a stationary independent winning strategy for $\mathsf{RC}_\tau^{\mathsf{fast}}(A)$.

Question: Are the above also equivalent with the following?

- *II* has a stationary independent winning strategy for $\mathsf{RC}_\tau^{\mathsf{closed}}(A)$.

Any counterexample (τ, A) is not projective and has size at least $\aleph_{\tau+1}$.