Congruence 5-permutability is not join prime

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Definition

A variety $\mathcal{V}$ is **congruence $n$-permutable** ($n \geq 2$) if every algebra $A \in \mathcal{V}$ satisfies $\alpha \circ^n \beta = \beta \circ^n \alpha$ for all congruences $\alpha, \beta \in \text{Con}(A)$.

- 5-permutability: $\alpha \circ \beta \circ \alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta \circ \alpha \circ \beta$.
- Congruence 2-permutability: $\alpha \circ \beta = \beta \circ \alpha$

Examples: groups, rings, varieties with a Maltsev-term:

$$m(x, y, y) \approx m(y, y, x) \approx x, \quad m(x, y, z) = xy^{-1}z,$$

$$(x, z) \in \alpha \circ \beta \Rightarrow x \alpha y \beta z \Rightarrow x \beta m(x, y, z) \alpha z \Rightarrow (x, z) \in \beta \circ \alpha$$

- Variety of lattices is not congruence $n$-permutable for any $n$:

$$(0, 1) \in \alpha \circ \beta \circ \alpha,$$

$$(0, 1) \notin \beta \circ \alpha \circ \beta.$$
\[ \alpha \lor \beta = \bigcup_n \alpha \circ^n \beta \text{ for any } \alpha, \beta \in \text{Con}(A) \]

\[ \alpha \lor \beta = \alpha \circ^n \beta \text{ in congruence } n\text{-permutable varieties} \]

- congruence \( n\)-permutability implies \( n+1\)-permutability

**Theorem (J. Hagemann, A. Mitschke; 1973)**

*For a variety \( \mathcal{V} \) and \( n \geq 2 \) the following are equivalent:*

- \( \mathcal{V} \) is congruence \( n\)-permutable,
- \( \varrho^{-1} \subseteq \varrho \circ^{n-1} \varrho \) for any \( A \in \mathcal{V} \) and reflexive relation \( \varrho \leq A^2 \),
- \( \mathcal{V} \) has ternary terms \( p_1, \ldots, p_{n-1} \) satisfying

\[
 x \approx p_1(x, y, y), \\
p_i(x, x, y) \approx p_{i+1}(x, y, y) \text{ for } 1 \leq i < n - 1, \\
p_{n-1}(x, x, y) \approx y.
\]

**Corollary**

\( \mathcal{V} \) is congruence \( n\)-permutable for some \( n \) if and only if every reflexive and transitive relation \( \varrho \leq A^2 \) of \( A \in \mathcal{V} \) is symmetric.
• $G$ is the variety of groups in the language $\cdot, -1, 1$
• $D_n$ is the variety of algebras having Hagemann-Mischke operations $p_1, \ldots, p_{n-1}$ for congruence $n$-permutability
• $BA$ is the variety of boolean algebras with $\lor, \land, ', 0, 1$
• $BR$ is the variety of boolean rings with $+, \cdot, 0, 1$

Interpretability: $D_2 \preceq G$, $BA \preceq BR \preceq BA$, $D_{n+1} \preceq D_n$

**Definition (W.D. Neumann, 1974)**

The variety $\mathcal{V}$ is **interpretable** in the variety $\mathcal{W}$ (notation $\mathcal{V} \preceq \mathcal{W}$) if for each $f$ $n$-ary basic operation of $\mathcal{V}$ there exists an $n$-ary term $t_f(x_1, \ldots, x_n)$ of $\mathcal{W}$ such that for each algebra $A = (A; F) \in \mathcal{W}$ the associated algebra $A' = (A; \{t_f \mid f \in F\})$ is in $\mathcal{V}$.

• constants: use unary operations satisfying $c(x) \approx c(y)$
• $\preceq$ is a quasiorder on the class of varieties
• equi-interpretability: $\mathcal{V} \equiv \mathcal{W}$ iff $\mathcal{V} \preceq \mathcal{W} \preceq \mathcal{V}$
Theorem

The class of varieties modulo equi-interpretability forms a bounded lattice (the lattice of interpretability types) with \( \overline{V} \lor \overline{W} = \overline{V \sqcup W} \) and \( \overline{V} \land \overline{W} = \overline{V \otimes W} \).

Definition

The coproduct of the varieties \( V = \text{Mod}(\Sigma) \) and \( W = \text{Mod}(\Delta) \) in disjoint languages is the variety \( V \sqcup W = \text{Mod}(\Sigma \cup \Delta) \).

Definition

The varietal product of \( V \) and \( W \) is the variety \( V \otimes W \) of algebras \( A \otimes B \) for \( A \in V \) and \( B \in W \) whose

- universe is \( A \times B \),
- basic operations are \( s \otimes t \) acting coordinate-wise for each pair of \( n \)-ary terms of \( V \) and \( W \).
O. Garcia, W. Taylor (1984): Lattice of interpretability types of varieties
- minimal element: sets (equi-interpretable with semigroups)
- maximal element: trivial algebras
- the class of idempotent varieties form a sublattice
- the class of finitely presented varieties forms a sublattice
- the class of varieties defined by linear equations forms a join sub-semilattice
- not modular
- meet prime elements: boolean algebras, lattices, semilattices
- meet irreducible elements: groups
- join prime elements: commutative groupoids, trivial algebras

J. Mycielski (1977): Lattice of interpretability types of first order theories
- local interpretability
- distributive
Some positive results:

- S. Tschantz (1983): congruence 2-permutability is join prime (unpublished)
- M. Valeriote, R. Willard (2014): congruence $n$-permutability is join-prime among idempotent varieties
- J. Opršal (2016): congruence $n$-permutability is join prime among varieties axiomatized by linear equations
- J. Opršal (2016); K. Kearnes, Á. Szendrei (2016): having an $n$-cube term is join prime among idempotent varieties
- L. Barto, J. Opršal, M. Pinsker (2018): congruence modularity is a prime filter among idempotent varieties

Some negative results:

- P. Marković, R. McKenzie (2008): having an $n$-ary near unanimity term is not join prime
- ...
Plan:

- Find two varieties $\mathcal{V}$ and $\mathcal{W}$ such that neither is $n$-permutable for any $n \geq 2$ but their coproduct is $n$-permutable for some $n$.
- $\mathcal{V}$ is not $n$-permutable for any $n$ if and only if it has an algebra $A \in \mathcal{V}$ and a compatible poset $\varrho \leq A^2$ which is not symmetric.
- Let $A'$ be the extension of $A$ with all order preserving operations of $\varrho$, and let $\mathcal{V}'$ be the variety generated by $A'$.
- $\mathcal{V}'$ and $\mathcal{W}'$ are still not $n$-permutable for any $n \geq 2$, but $\mathcal{V} \leq \mathcal{V}'$ and $\mathcal{W} \leq \mathcal{W}'$ so their coproduct is more likely to be $n$-permutable for some $n$.
- We need to search for posets.
- Need to understand algebras in the variety defined by a poset.
- We need to understand congruences, compatible quasiorders, reflexive relations in these varieties and in their coproduct.

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Definition

Let $\mathbb{P} = (P; \leq)$ be a poset. The clone $\text{Pol} (\mathbb{P})$ of polymorphisms of $\mathbb{P}$ is the ranked set of order preserving maps $f : \mathbb{P}^n \to \mathbb{P}$.

- Let $\mathbb{P} = (\{0, 1\}; \leq), \mathbf{P} = (P; \text{Pol}(\mathbb{P}))$ and $\mathcal{V} = \text{HSP}(\mathbf{P})$
- $\wedge, \vee, 0, 1 \in \text{Pol}(\mathbb{P})$ and these operations generate the clone
- $\mathbf{P}$ is term equivalent with the two-element bounded distributive lattice
- $\mathcal{V}$ is equi-interpretable with the variety of bounded distributive lattices
- $\mathcal{V}$ is locally finite (finitely generated free algebras are finite)
- For each finite algebra $\mathbf{A} \in \mathcal{V}$ there is a finite quasiorder $\mathcal{Q}$ such that $\mathbf{A} = \mathbb{P}^\mathcal{Q}$ with point-wise ops (Priestley-duality)
- What are the congruences, compatible quasiorder, compatible reflexive relations of $\mathbf{A}$?
Theorem

Let $\mathcal{P}$ be a finite bounded poset with a compatible near-unanimity operation, and $\mathcal{P}$ be the corresponding finitely presented variety. Let $\mathcal{M}$ be any variety defined by a linear Maltsev-condition that is not already satisfied by $\mathcal{P}$.

1. Then $\mathcal{P} \amalg \mathcal{M}$ is congruence $n$-permutable for some $n \geq 2$.
2. If $\mathcal{P}$ is the 6-element poset with order $0 \leq a, b \leq c, d \leq 1$, and $\mathcal{M} = \text{Mod}(m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x)$, then $\mathcal{P} \amalg \mathcal{M}$ is congruence 5-permutable.

Corollary

In the lattice of interpretability types

1. congruence $n$-permutability for some $n \geq 2$ is not a prime filter,
2. congruence 5-permutability is not a join prime element.
Proof sketch of first result:

- Let $\mathcal{P} = (P; \leq)$ be the 6-element poset

  $\begin{array}{c}
  1 \\
  c \\
  a \\
  0 \\
  0 \\
  c \\
  b \\
  1
  \end{array}$

- $\mathcal{P}$ has a compatible 5-ary near-unanimity operation
- Baker-Pixley: $\text{Pol}(\mathcal{P})$ is finitely generated by $p_1, \ldots, p_k$
- Let $\mathcal{P}$ be the variety generated by $\mathcal{P} = (P; p_1, \ldots, p_k)$
- $\mathcal{P}$ is congruence distributive, does not have a majority term $m$
- $\mathcal{P}$ is simple, has no non-trivial subalgebras, no other SI’s in $\mathcal{P}$
- Let $\mathcal{M}$ be the variety of algebras with majority operation

  $m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x$

- Take $A \in \mathcal{P} \sqcup \mathcal{M}$ and $\varrho \in A^2$ that is reflexive and transitive, need to show that $\varrho$ is symmetric
Take a failure of $n$-permutability, e.g. $(f, g) \in \varrho \setminus \varrho^{-1}$

Let $B_0 = Sg_P(\{f, g\})$, $B_1 = Sg_P(\{m(x, y, z) \mid x, y, z \in B_1\})$

$B_0$ and $B_1$ are finite algebras in $\mathcal{P}$

$B_1 \leq_{sd} P^R$ for some finite set $R$

$B_1$ is in the relational clone generated by $P$, so it is defined by a primitive positive formula with free variable set $R$.

There exists a poset $Q_1 = (Q; q_1)$ such that $Q \supseteq R$ and $B_1 = \left.P^Q_{\mid R} \right.$ is the set of order preserving functions from $Q_1$ to $P$

There is a quasi-order $Q_0 = (Q; q_0)$ such that $B_1 = \left. P^Q_{\mid R} \right.$

Since $B_0 \leq B_1$ we have $q_0 \supseteq q_1$

Projection congruences: $\eta_r = \{ (u, v) \mid u(r) = v(r) \}$ for $r \in R$

Every congruence of $B_0$ and $B_1$ are product congruences, i.e., the intersection of a set of projection congruences

$\varrho_0 = \varrho\mid_{B_0}$, $\varrho_1 = \varrho\mid_{B_1}$ are compatible quasiorders of $B_0$ and $B_1$

We argue, that $\varrho_0$ and $\varrho_1$ are product quasiorders
Definition

The set of compatible quasiorders of an algebra $A$ is

$$\text{Quo}(A) = \{ \alpha \leq A^2 \mid \alpha \text{ is reflexive and transitive} \}.$$  

- $\text{Quo}(A)$ forms an (involution) lattice with $\alpha \wedge \beta = \alpha \cap \beta$ and $\alpha \vee \beta = \overline{\alpha \cup \beta}$, where $\overline{\alpha \cup \beta}$ is the transitive closure of $\alpha \cup \beta$.
- The set $\text{Con}(A)$ of congruences forms a sublattice of $\text{Quo}(A)$.

Theorem (G. Gyenizse, M. M; 2018)

1. A locally finite variety $V$ is congruence distributive if and only if it is quasiorder distributive.
2. A locally finite variety is congruence modular if and only if it is quasiorder modular.
3. The variety of semilattices is not quasiorder meet semi-distributive (but it is congruence meet semi-distributive).
4. For a finite algebra $A$ in a congruence meet semi-distributive variety $\text{Quo}(A)$ has no sublattice isomorphic to $M_3$. 
Projection quasiorders: for each \( r \in R \)

\[
\sigma_r = \{ (u, v) \mid u(r) \leq v(r) \}
\]

\[
\tau_r = \{ (u, v) \mid u(r) \geq v(r) \}
\]

\( \eta_r = \sigma_r \land \tau_r \)

There are \( S, T \subseteq R \) such that \( \varrho_1 = (\land_{s \in S} \sigma_s) \land (\land_{t \in T} \tau_t) \)

\( (g, f) \not\in \varrho_1 \), so we can choose \( s \in S \setminus T \) such that \( g(s) \not\leq f(s) \)

The elements \( a, b, c, d \) exhibit the failure of not having a majority term: \( a, b \leq m(a, b, c) \leq c, d \) must hold, but there is no such element \( m(a, b, c) \)

Find elements \( u_a, u_b, u_c, u_d \) that exhibit this behavior in \( B_0 \) at \( s \): in the \( q_0 \)-block of \( s \), \( u_x \) takes value \( x \), above it it takes 1 and everywhere else it takes 0

\( u_a, u_b \sigma_s u_c, u_d \) holds in \( B_0 \)

Thus \( u_a, u_b \sigma_s m(u_a, u_b, u_c) \sigma_s u_c, u_d \) must hold in \( B_1 \)

There is no such element because of coordinate \( s \), a contradiction.
Thank You!