Congruence 5-permutability is not join prime

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Definition

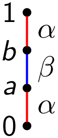
A variety $\mathcal{V}$ is **congruence n-permutable** ($n \geq 2$) if every algebra $\mathbf{A} \in \mathcal{V}$ satisfies $\alpha \circ^n \beta = \beta \circ^n \alpha$ for all congruences $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$.

- 5-permutability: $\alpha \circ \beta \circ \alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta \circ \alpha \circ \beta$.
- Congruence 2-permutability: $\alpha \circ \beta = \beta \circ \alpha$
  Examples: groups, rings, varieties with a Maltsev-term:

$$m(x, y, y) \approx m(y, y, x) \approx x, \qquad m(x, y, z) = xy^{-1}z,$$

$$(x, z) \in \alpha \circ \beta \Rightarrow x \; \alpha \; y \; \beta \; z \Rightarrow x \; \beta \; m(x, y, z) \; \alpha \; z \Rightarrow (x, z) \in \beta \circ \alpha$$

- Variety of lattices is not congruence n-permutable for any n:



$$\begin{aligned} &\alpha = \{(0, a), (b, 1), \dots\}, \\ &\beta = \{(a, b), \dots\}, \\ &(0, 1) \in \alpha \circ \beta \circ \alpha, \\ &(0, 1) \notin \beta \circ \alpha \circ \beta. \end{aligned}$$

- $\alpha \vee \beta = \bigcup_n \alpha \circ^n \beta$ for any $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$
- $\alpha \vee \beta = \alpha \circ^n \beta$ in congruence n-permutable varieties
- congruence n-permutability implies $n+1$-permutability

Theorem (J. Hagemann, A. Mitschke; 1973)

For a variety $\mathcal{V}$ and $n \geq 2$ the following are equivalent:

- *$\mathcal{V}$ is congruence n-permutable,*
- *$\varrho^{-1} \subseteq \varrho \circ^{n-1} \varrho$ for any $\mathbf{A} \in \mathcal{V}$ and reflexive relation $\varrho \leq \mathbf{A}^2$,*
- *$\mathcal{V}$ has ternary terms $p_1, \ldots, p_{n-1}$ satisfying*

$$\begin{aligned} x &\approx p_1(x,y,y), \\ p_i(x,x,y) &\approx p_{i+1}(x,y,y) \text{ for } 1 \leq i < n-1, \\ p_{n-1}(x,x,y) &\approx y. \end{aligned}$$

Corollary

$\mathcal{V}$ is congruence n-permutable for some n if and only if every reflexive and transitive relation $\varrho \leq \mathbf{A}^2$ of $\mathbf{A} \in \mathcal{V}$ is symmetric.

- $\mathcal{G}$ is the variety of groups in the language $\cdot, ^{-1}, 1$
- $\mathcal{D}_n$ is the variety of algebras having Hagemann-Mischke operations $p_1, \ldots, p_{n-1}$ for congruence n-permutability
- $\mathcal{BA}$ is the variety of boolean algebras with $\vee, \wedge, ', 0, 1$
- $\mathcal{BR}$ is the variety of boolean rings with $+, \cdot, 0, 1$

Interpretability: $\mathcal{D}_2 \preceq \mathcal{G}$, $\mathcal{BA} \preceq \mathcal{BR} \preceq \mathcal{BA}$, $\mathcal{D}_{n+1} \preceq \mathcal{D}_n$

Definition (W.D. Neumann, 1974)

The variety $\mathcal{V}$ is **interpretable** in the variety $\mathcal{W}$ (notation $\mathcal{V} \preceq \mathcal{W}$) if for each f n-ary basic operation of $\mathcal{V}$ there exists an n-ary term $t_f(x_1, \ldots, x_n)$ of $\mathcal{W}$ such that for each algebra $\mathbf{A} = (A; \mathcal{F}) \in \mathcal{W}$ the associated algebra $\mathbf{A}' = (A; \{t_f \mid f \in \mathcal{F}\})$ is in $\mathcal{V}$.

- constants: use unary operations satisfying $c(x) \approx c(y)$
- $\preceq$ is a quasiorder on the class of varieties
- equi-interpretability: $\mathcal{V} \equiv \mathcal{W}$ iff $\mathcal{V} \preceq \mathcal{W} \preceq V$

Theorem

The class of varieties modulo equi-interpretability forms a bounded lattice (the lattice of **interpretability types***) with* $\overline{\mathcal{V}} \vee \overline{\mathcal{W}} = \overline{\mathcal{V} \amalg \mathcal{W}}$ *and* $\overline{\mathcal{V}} \wedge \overline{\mathcal{W}} = \overline{\mathcal{V} \otimes \mathcal{W}}$.

Definition

The **coproduct** of the varieties $\mathcal{V} = \mathrm{Mod}(\Sigma)$ and $\mathcal{W} = \mathrm{Mod}(\Delta)$ in disjoint languages is the variety $\mathcal{V} \amalg \mathcal{W} = \mathrm{Mod}(\Sigma \cup \Delta)$.

Definition

The **varietal product** of $\mathcal{V}$ and $\mathcal{W}$ is the variety $\mathcal{V} \otimes \mathcal{W}$ of algebras $\mathbf{A} \otimes \mathbf{B}$ for $\mathbf{A} \in \mathcal{V}$ and $\mathbf{B} \in \mathcal{W}$ whose

- universe is $A \times B$,
- basic operations are $s \otimes t$ acting coordinate-wise for each pair of n-ary terms of $\mathcal{V}$ and $\mathcal{W}$.

O. Garcia, W. Taylor (1984): Lattice of interpretability types of varieties

- minimal element: sets (equi-interpretable with semigroups)
- maximal element: trivial algebras
- the class of idempotent varieties form a sublattice
- the class of finitely presented varieties forms a sublattice
- the class of varieties defined by linear equations forms a join sub-semilattice
- not modular
- meet prime elements: boolean algebras, lattices, semilattices
- meet irreducible elements: groups
- join prime elements: commutative groupoids, trivial algebras

J. Mycielski (1977): Lattice of interpretability types of first order theories

- local interpretability
- distributive

Some positive results:

- S. Tschantz (1983): congruence 2-permutability is join prime (unpublished)
- M. Valeriote, R. Willard (2014): congruence n-permutability is join-prime among idempotent varieties
- J. Opršal (2016): congruence n-permutability is join prime among varieties axiomatized by linear equations
- J. Opršal (2016); K. Kearnes, Á. Szendrei (2016): having an n-cube term is join prime among idempotent varieties
- L. Barto, J. Opršal, M. Pinsker (2018): congruence modularity is a prime filter among idempotent varieties

Some negative results:

- P. Marković, R. McKenzie (2008): having an n-ary near unanimity term is not join prime
- ...

Plan:

- Find two varieties $\mathcal{V}$ and $\mathcal{W}$ such that neither is n-permutable for any $n \geq 2$ but their coproduct is n-permutable for some n.
- $\mathcal{V}$ is not n-permutable for any n if and only if it has an algebra $\mathbf{A} \in \mathcal{V}$ and a compatible poset $\varrho \leq \mathbf{A}^2$ which is not symmetric.
- Let $\mathbf{A}'$ be the extension of $\mathbf{A}$ with all order preserving operations of ϱ, and let $\mathcal{V}'$ be the variety generated by $\mathbf{A}'$.
- $\mathcal{V}'$ and $\mathcal{W}'$ are still not n-permutable for any $n \geq 2$, but $\mathcal{V} \preceq \mathcal{V}'$ and $\mathcal{W} \preceq \mathcal{W}'$ so their coproduct is more likely to be n-permutable for some n.
- We need to search for posets.
- Need to understand algebras in the variety defined by a poset.
- We need to understand congruences, compatible quasiorders, reflexive relations in these varieties and in their coproduct.

Definition

Let $\mathbb{P} = (P; \leq)$ be a poset. The clone $\mathrm{Pol}(\mathbb{P})$ of **polymorphisms** of $\mathbb{P}$ is the ranked set of order preserving maps $f : \mathbb{P}^n \to \mathbb{P}$.

- Let $\mathbb{P} = (\{0,1\}; \leq)$, $\mathbf{P} = (P; \mathrm{Pol}(\mathbb{P}))$ and $\mathcal{V} = \mathrm{HSP}(\mathbf{P})$
- $\wedge, \vee, 0, 1 \in \mathrm{Pol}(\mathbb{P})$ and these operations generate the clone
- $\mathbf{P}$ is term equivalent with the two-element bounded distributive lattice
- $\mathcal{V}$ is equi-interpretable with the variety of bounded distributive lattices
- $\mathcal{V}$ is locally finite (finitely generated free algebras are finite)
- For each finite algebra $\mathbf{A} \in \mathcal{V}$ there is a finite quasiorder $\mathbb{Q}$ such that $\mathbf{A} = \mathbb{P}^{\mathbb{Q}}$ with point-wise ops (Priestley-duality)
- What are the congruences, compatible quasiorder, compatible reflexive relations of $\mathbf{A}$?

Theorem

Let $\mathbb{P}$ be a finite bounded poset with a compatible near-unanimity operation, and $\mathcal{P}$ be the corresponding finitely presented variety. Let $\mathcal{M}$ be any variety defined by a linear Maltsev-condition that is not already satisfied by $\mathcal{P}$.

1. *Then $\mathcal{P} \amalg \mathcal{M}$ is congruence n-permutable for some $n \geq 2$.*
2. *If $\mathbb{P}$ is the 6-element poset with order $0 \leq a, b \leq c, d \leq 1$, and $\mathcal{M} = \mathrm{Mod}(m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x)$, then $\mathcal{P} \amalg \mathcal{M}$ is congruence 5-permutable.*

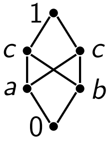
Corollary

In the lattice of interpretability types

1. *congruence n-permutability for some $n \geq 2$ is not a prime filter,*
2. *congruence 5-permutability is not a join prime element.*

Proof sketch of first result:

- Let $\mathbb{P} = (P; \leq)$ be the 6-element poset



- $\mathbb{P}$ has a compatible 5-ary near-unanimity operation
- Baker-Pixley: $\mathrm{Pol}(\mathbb{P})$ is finitely generated by $p_1, \ldots p_k$
- Let $\mathcal{P}$ be the variety generated by $\mathbf{P} = (P; p_1, \ldots, p_k)$
- $\mathcal{P}$ is congruence distributive, does not have a majority term m
- $\mathbf{P}$ is simple, has no non-trivial subalgebras, no other SI's in $\mathcal{P}$
- Let $\mathcal{M}$ be the variety of algebras with majority operation

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x$$

- Take $\mathbf{A} \in \mathcal{P} \amalg \mathcal{M}$ and $\varrho \in \mathbf{A}^2$ that is reflexive and transitive, need to show that ϱ is symmetric

- Take a failure of n-permutability, e.g. $(f, g) \in \varrho \setminus \varrho^{-1}$
- Let $\mathbf{B}_0 = \mathrm{Sg}_{\mathcal{P}}(\{f, g\})$, $\mathbf{B}_1 = \mathrm{Sg}_{\mathcal{P}}(\{m(x, y, z) \mid x, y, z \in B_1\})$
- $\mathbf{B}_0$ and $\mathbf{B}_1$ are finite algebras in $\mathcal{P}$
- $\mathbf{B}_1 \leq_{\mathrm{sd}} \mathbf{P}^R$ for some finite set R
- B_1 is in the relational clone generated by $\mathbb{P}$, so it is defined by a primitive positive formula with free variable set R.
- There exists a poset $\mathbb{Q}_1 = (Q; q_1)$ such that $Q \supseteq R$ and $B_1 = \mathbb{P}^{\mathbb{Q}_1}|_R$ is the set of order preserving functions from $\mathbb{Q}_1$ to $\mathbb{P}$
- There is a quasi-order $\mathbb{Q}_0 = (Q; q_0)$ such that $B_1 = \mathbb{P}^{\mathbb{Q}_1}$
- Since $\mathbf{B}_0 \leq \mathbf{B}_1$ we have $q_0 \supseteq q_1$
- Projection congruences: $\eta_r = \{\, (u, v) \mid u(r) = v(r)\,\}$ for $r \in R$
- Every congruence of $\mathbf{B}_0$ and $\mathbf{B}_1$ are product congruences, i.e., the intersection of a set of projection congruences
- $\varrho_0 = \varrho|_{B_0}$, $\varrho_1 = \varrho|_{B_1}$ are compatible quasiorders of $\mathbf{B}_0$ and $\mathbf{B}_1$
- We argue, that ϱ_0 and ϱ_1 are product quasiorders

Definition

The set of **compatible quasiorders** of an algebra $\mathbf{A}$ is

$$\mathrm{Quo}(\mathbf{A}) = \{\, \alpha \leq \mathbf{A}^2 \mid \alpha \text{ is reflexive and transitive} \,\}.$$

- $\mathrm{Quo}(\mathbf{A})$ forms an (involution) lattice with $\alpha \wedge \beta = \alpha \cap \beta$ and $\alpha \vee \beta = \overline{\alpha \cup \beta}$, where $\overline{\alpha \cup \beta}$ is the transitive closure of $\alpha \cup \beta$.
- The set $\mathrm{Con}(\mathbf{A})$ of congruences forms a sublattice of $\mathrm{Quo}(\mathbf{A})$.

Theorem (G. Gyenizse, M. M; 2018)

1. *A locally finite variety $\mathcal{V}$ is congruence distributive if and only if it is quasiorder distributive*
2. *A locally finite variety is congruence modular if and only if it is quasiorder modular.*
3. *The variety of semilattices is not quasiorder meet semi-distributive (but it is congruence meet semi-distributive).*
4. *For a finite algebra* $\mathbf{A}$ *in a congruence meet semi-distributive variety* $\mathrm{Quo}(\mathbf{A})$ *has no sublattice isomorphic to* $\mathbf{M}_3$.

- Projection quasiorders: for each $r \in R$

$$\sigma_r = \{ (u, v) \mid u(r) \leq v(r) \}$$
$$\tau_r = \{ (u, v) \mid u(r) \geq v(r) \}$$

- $\eta_r = \sigma_r \wedge \tau_r$
- There are $S, T \subseteq R$ such that $\varrho_1 = (\bigwedge_{s \in S} \sigma_s) \wedge (\bigwedge_{t \in T} \tau_t)$
- $(g, f) \notin \varrho_1$, so we can choose $s \in S \setminus T$ such that $g(s) \not\leq f(s)$
- The elements a, b, c, d exhibit the failure of not having a majority term: $a, b \leq m(a, b, c) \leq c, d$ must hold, but there is no such element $m(a, b, c)$
- Find elements u_a, u_b, u_c, u_d that exhibit this behavior in $\mathbf{B}_0$ at s: in the q_0-block of s, u_x takes value x, above it it takes 1 and everywhere else it takes 0
- $u_a, u_b \; \sigma_s \; u_c, u_d$ holds in $\mathbf{B}_0$
- Thus $u_a, u_b \; \sigma_s \; m(u_a, u_b, u_c) \; \sigma_s \; u_c, u_d$ must hold in $\mathbf{B}_1$
- There is no such element because of coordinate s, a contradiction.

Thank You!