Relation algebras of Sugihara, Belnap, Meyer, Church. Relevance logic of Tarski.

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Old and New Associations

1. Historical association: Names — Topic

   Sugihara, Belnap, Meyer, Church — Relevance Logic

   Tarski — Relation Algebras

2. Relevance logic algebras of Sugihara, Belnap, Meyer, Church turn out to be definitional reducts of proper relation algebras
3. The provable-with-four-variables characterization of Tarski's equational theory of relation algebras yields a new relevance logic
4. By 2 and 3, the association can be inverted—

   Belnap, Meyer, Sugihara, Church — Proper relation algebras

   Tarski — A relevance logic

Relation Algebras [1987, §8.2]

To get a relation algebra (RA)—

1. Start with a Boolean algebra $\langle B, +, \cdot, ^{-}, 0, 1 \rangle$
2. Add a binary operation $\;;$
3. Add a unary operation $\;\breve{}$
4. Add a constant $\;1'$
5. Insist that Tarski's Ten Axioms [1987] hold

$$\langle B, +, \cdot, ^{-}, 0, 1, \;;\;, \breve{}\;, 1' \rangle \models \text{(Ra I)}, \cdots, \text{(Ra X)}$$

6. The associative law (Ra IV) $\quad x;(y;z) = (x;y);z$

Proper Relation Algebras [1987, §8.3]

To get a proper relation algebra (PRA)—

1. Start with a set U, or an equivalence relation E
2. Use all binary relations on U, or all subsets of E
3. Add union $\cup$ and intersection $\cap$
4. Add complementation $^-$ (with respect to $U \times U$ or E)
5. Add $|$, composition $R|S$ of relations R, S
6. Add $^{-1}$, converse R^{-1} of R
7. Add constants $U \times U$ or E, $\emptyset$, the identity relation on U $\mathrm{Id}_U = \{\langle u, u\rangle : u \in U\}$ or the identity part of E $\{\langle u, u\rangle : \langle u, u\rangle \in E\}$
8. Take a subalgebra
9. Representable RA means isomorphic to a PRA

Characterizations of Equations True in PRA, RA [1987]

1. (Tarski 1940s) Every RA equation translates into a sentence of first-order logic (of binary relations) using at most 3 variables, and conversely
2. (Tarski 1940s) The associative law for $|$ requires 4 variables to prove, but the other axioms only need 3
3. (Tarski 1940s) An equation is valid in every PRA iff its translation can be proved in first-order logic
4. [1978] An equation is valid in every relation algebra iff it can be proved in first-order logic restricted to 4 variables — RA=provable-with-four-variables motivates a definition:
5. [3/2019] A formula (using $\wedge$, $\vee$, $\rightarrow$, $\sim$) is in Tarski's relevance logic if it can be proved in first-order logic restricted to 4 variables

Definitional Subreducts

To get a definitional subreduct —

1. Start with an algebra (a set with some operations)
2. Add some defined operations
3. Delete some operations
4. Take a subalgebra with respect to the remaining operations

Definitional Subreducts of PRAs [1/2019]

To represent algebras of relevance logic with binary relations by the First Method

1. Start with a proper relation algebra of relations on U

$$\langle \wp(U \times U), \cup, \cap, ^{-}, \emptyset, U \times U, |, ^{-1}, \mathsf{Id}_U \rangle$$
$$R|S = \{\langle x, z\rangle : \exists y(\langle x, y\rangle \in R \wedge \langle y, z\rangle \in S)\}$$
$$R^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in R\}$$
$$\mathsf{Id}_U = \{\langle u, u\rangle : u \in U\}$$

2. Add $\quad R \to S = \overline{R^{-1}|\overline{S}} \qquad {\sim}R = \overline{R^{-1}}$
3. Delete $\quad ^{-} \quad \emptyset \quad U \times U \quad | \quad ^{-1} \quad \mathsf{Id}_U$
4. Take subalgebra wrt $\quad \cup \quad \cap \quad \to \quad \sim$

Definitional Subreducts of PRAs [1/2019]

To represent algebras of relevance logic with binary relations by the Second Method, relativize to diversity

1. Start with a proper relation algebra of relations on U

$$\left\langle \wp(U \times U), \cup, \cap, {}^{-}, \emptyset, U \times U, |, {}^{-1}, \mathrm{Id}_U \right\rangle$$

2. Add $\quad R \to' S = \overline{R^{-1}|\overline{S}} \cap \overline{\mathrm{Id}_U} \qquad \sim' R = \overline{R^{-1}} \cap \overline{\mathrm{Id}_U}$
3. Delete $\quad {}^{-} \quad \emptyset \quad U \times U \quad | \quad {}^{-1} \quad \mathrm{Id}_U$
4. Take subalgebra wrt $\quad \cup \quad \cap \quad \to' \quad \sim'$

Algebras of Relevance Logic are Definitional Subreducts of PRAs [2007][2010][1/2019]

Examples of algebras used in relevance logic that turn out to be definitional subreducts of proper relation algebras—

1. Belnap [1960][1975][1982]: the M_0 matrices (8-element algebra)
2. Meyer [1982][2003]: the crystal matrices (6-element algebra)
3. Church [1982][2003]: the diamond matrices (4-element algebra)
4. Meyer [1975][1982]: the RM84 matrices (8-element algebra)
5. Sugihara [1955][1975]: the odd and even matrices (algebras of every finite size, two infinite ones) — complete for **RM**

Belnap's Relation Algebra [2007][2010]

1. Start with the proper relation algebra of all relations on $U = \mathbb{Q}$, the set of rational numbers
2. Belnap's relation algebra $\mathfrak{S}_{\{0\}}$ is the subalgebra whose elements are these binary relations on $\mathbb{Q}$:

   $$\emptyset \qquad = \qquad < \qquad > \qquad \leq \qquad \geq \qquad \neq \qquad \mathbb{Q} \times \mathbb{Q}$$

3. Belnap's RA $\mathfrak{S}_{\{0\}}$ has been called The Point Algebra since the 1980s because its elements are all the ways two points on the rational line can be related to each other
4. The algebra called Belnap's M_0 matrices is a definitional reduct of Belnap's RA $\mathfrak{S}_{\{0\}}$ by the First Method

Meyer's Crystal Relation Algebra [1/2019]

1. Start with the proper relation algebra of all relations on $U = \mathbb{Q}^2$, the set of pairs of rational numbers
2. Meyer's crystal relation algebra $\mathfrak{S}_{\{0,1\}}$ is the subalgebra generated by those relations whose images of $\langle 0, 0\rangle$ are the positive and negative x-axes, and the upper and lower half planes (with x-axis excluded)
3. Meyer's crystal matrices occur as a 6-element definitional subreduct of Meyer's RA $\mathfrak{S}_{\{0,1\}}$ using the Second Method
4. Meyer's RA $\mathfrak{S}_{\{0,1\}}$ is isomorphic to RA 2_{83} in [2006]

Church's Diamond Relation Algebra [1/2019]

1. Start with the proper relation algebra of all relations on a set U with 6 elements
2. Church's diamond relation algebra 2_7 is the subalgebra generated by the equivalence relation with exactly two equivalence classes, three elements in each class
3. The algebra called Church's diamond is a 4-element definitional subreduct of Church's RA 2_7 by the Second Method
4. Church's RA 2_7 has size 8, and is representable on sets of size 6 or more; see [2006]

Meyer's RM84 Relation Algebra [2010]

1. Start with the proper relation algebra of all relations on $U = \{0, 1, 2, 3, 4, 5, 6\}$
2. Meyer's RM84 relation algebra 3_3 is the subalgebra generated by the relations $\{\langle n, n+i \mod 7\rangle : n \in U,\ i = 1, 2, 4\}$ and $\{\langle n, n+i \mod 7\rangle : n \in U,\ i = 3, 5, 6\}$
3. The algebra called Meyer's RM84 in [1982] (unnamed in [1975]) is a definitional reduct of Meyer's RA 3_3 by the First Method
4. Meyer's RA 3_3 has size 8 and is representable . . .
   . . . on sets with 7 elements (uniquely),
   . . . on every set with more than 8 elements (not uniquely),
   . . . but not on 8 elements; see [2006]

Sugihara's Relation Algebras [1/2019]

1. Start with the proper relation algebra of all relations on U_I, for any $I \subseteq \mathbb{Z}$, the $\mathbb{Z}$-indexed sequences of rationals that are zero outside I and eventually always zero:

$$U_I = \{q \in {}^{\mathbb{Z}}\mathbb{Q} : (\forall i \in \mathbb{Z} \setminus I)(q_i = 0) \wedge (\exists n \in \mathbb{Z})(\forall i > n)(q_i = 0)\}$$

2. Sugihara's relation algebra $\mathfrak{S}_I$ is the subalgebra completely generated (use arbitrary unions) by the relations L^I_n, $n \in I$, "less-than" at n (the last place they differ)

$$L^I_n = \{\langle q, r\rangle : q, r \in U_I,\ q_n < r_n \text{ and } q_i = r_i \text{ for all } i > n\}$$

3. Examples: Belnap's RA $\mathfrak{S}_{\{0\}}$ and Meyer's RA $\mathfrak{S}_{\{0,1\}}$
4. Defined for finite I in [2010]

The Even Sugihara Matrices [1/2019]

1. The 2- and 4-element Sugihara algebras are definitional subreducts of Belnap's RA $\mathfrak{S}_{\{0\}}$ by the First Method [2007][2010]
2. Every Sugihara algebra of even cardinality is a definitional subreduct of Sugihara's RA $\mathfrak{S}_{\mathbb{Z}}$ by the First Method
3. The algebra originally defined in Sugihara [1955] is a definitional subreduct of Sugihara's RA $\mathfrak{S}_{\mathbb{Z}}$ by the First Method

The Odd Sugihara Matrices [1/2019]

1. The Sugihara matrices of odd cardinality have an element fixed by $\sim$ (it is its own negation)
2. The 1- and 3-element Sugihara algebras are definitional subreducts of Belnap's RA $\mathfrak{S}_{\{0\}}$ by the Second Method
3. The 5-element Sugihara algebra is a definitional subreduct of Meyer's RA $\mathfrak{S}_{\{0,1\}}$ by the Second Method
4. Every Sugihara algebra of odd cardinality is a definitional subreduct of Sugihara's RA $\mathfrak{S}_{\mathbb{Z}}$ by the Second Method

Meyer's Completeness Theorem for **RM** [1975]

Significance of Sugihara matrices/algebras —

1. Every Sugihara algebra is a linearly ordered chain, and its designated values are the top half
2. When a Sugihara algebra is a subreduct of a PRA, the designated values are the relations that contain the identity relation [2010][1/2019]
3. (R.K.Meyer) the Sugihara matrices are characteristic for the Dunn-McCall system of logic called **RM** or **R**-mingle, i.e.,
4. (R.K.Meyer) A formula is a theorem of **RM** iff it is mapped to a designated value by every homomorphism from the formula algebra into a Sugihara algebra

Combined Completeness Theorem for **RM** [2010][1/2019]

A class of algebras of binary relations, characteristic for **RM**—

Definition

1. $K_{\mathbf{RM}}$ is the class of algebras $\mathfrak{K} = \langle K, \cup, \cap, \to, \sim \rangle$, where
    1.1 K is a set of <u>dense</u> and <u>transitive</u> binary relations on a set U
    1.2 K is closed under $\cup$, $\cap$, $\to$, and $\sim$
    1.3 K is <u>commutative</u> under $|$
2. A formula is valid in $K_{\mathbf{RM}}$ if it is sent to a relation containing the identity relation by every homomorphism from the algebra of formulas into an algebra in $K_{\mathbf{RM}}$

Theorem ($K_{\mathbf{RM}}$ characterizes **RM**)

RM *is the set of formulas valid in* $K_{\mathbf{RM}}$

Classification of Axioms and Rules of **RM** [1/2019]

Axioms of **RM** that are valid for all binary relations A, B, and C

$$A \to A \tag{A1}$$

$$A \wedge B \to A \tag{A5}$$

$$A \wedge B \to B \tag{A6}$$

$$(A \to B) \wedge (A \to C) \to (A \to B \wedge C) \tag{A7}$$

$$A \to A \vee B \tag{A8}$$

$$B \to A \vee B \tag{A9}$$

$$(A \to C) \wedge (B \to C) \to (A \vee B \to C) \tag{A10}$$

$$A \wedge (B \vee C) \to (A \wedge B) \vee C \tag{A11}$$

$$\sim\sim A \to A \tag{A13}$$

$$A, B \vdash A \wedge B \tag{R1}$$

$$A \to B, A \vdash B \tag{R2}$$

Classification of Axioms of **RM** [1/2019]

Axioms that are valid…

1. …if all relations commute under $|$:

$$(A \to B) \to ((B \to C) \to (A \to C)) \tag{A2}$$

$$A \to ((A \to B) \to B) \tag{A3}$$

2. …if and only if all relations are dense:

$$(A \to (A \to B)) \to (A \to B) \tag{A4}$$

3. …if and only if all relations commute under $|$:

$$(A \to {\sim}B) \to (B \to {\sim}A) \tag{A12}$$

4. …if and only if all relations are transitive:

$$A \to (A \to A) \tag{A14}$$

Tarski's Relevance Logic [3/2019]

Tarski's Relevance Logic ...

1. ... includes axioms and rules of **RM** that are valid for all binary relations A, B, and C:
   A1, A5, A6, A7, A8, A9, A10, A11, A13, R1, R2.
2. ... excludes the axioms involving density, commutativity, and transitivity: A2, A3, A4, A12, A14
3. ... includes (*), where $A \circ B = \sim(A \to \sim B)$,

$$(A \circ B) \wedge C \to ((A \wedge \sim D) \circ B) \vee (A \circ (B \wedge (D \circ C))) \quad (*)$$

4. (*) is not in the Anderson-Belnap system **R**, nor in **RM**,
5. (*) is a counterexample to a theorem of Kowalski 2012, that the class of commutative dense RAs is characteristic for **R**

A Chronological Bibliography, Part 1 of 2

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Abstract. Algebras of Sugihara, Belnap, Meyer, Church

- Sugihara's relation algebra is a proper relation algebra containing chains of relations isomorphic to Sugihara's original matrices.
- Belnap's matrices form a definitional reduct of a proper relation algebra known as the Point Algebra.
- The 2-element and 4-element Sugihara matrices are definitional subreducts of the Point Algebra.
- Meyer's crystal matrices, Meyer's RM84 matrices, and Church's diamond matrices are also definitional subreducts of proper relation algebras.
- The representation of Sugihara matrices as algebras of binary relations, together with Meyer's characterization of R-mingle by Sugihara matrices, yields the characterization of R-mingle by sets of transitive dense relations that commute under composition.
- The axioms of the Dunn-McCall system R-mingle can therefore be understood as statements about binary relations.

Abstract. Relevance Logic of Tarski

- The definition of Tarski's relevance logic arises from his work on first-order logic restricted to finitely many variables.
- It is inspired by the characterization of equations true in all relation algebras as those provable in first-order logic restricted to 4 variables.
- It has Belnap's variable-sharing property and avoids the paradoxes of implication.
- It contains the Basic Logic of Routley, Plumwood, Meyer, Brady, along with several derived rules of inference.
- It does not include several formulas used as axioms in the Anderson Belnap system R, such as Contraposition.
- It contains a formula (outside both R and R-mingle) that is a counterexample to a completeness theorem of Kowalski (that the system R is complete with respect to the class of dense commutative relation algebras).