On the structure of idempotent residuated lattices

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Outline

- Monoidal preorder of idempotent residuated lattices
- Idempotent residuated chains
- Conservative residuated lattices and catalan algebras
- Amalgamation property for semilinear idempotent RLs
- Some results about idempotent involutive residuated lattices

Introduction

A **residuated poset** is a partially ordered algebra $(A, \leqslant, \cdot, \mathbf{1}, /, \backslash)$ such that

- $(A, \leqslant)$ is a **poset**,
- $(A, \cdot, \mathbf{1})$ is a **monoid** and
- $/, \backslash$ are the **right** and **left residuals** of $\cdot$, i.e., the residuation property

$$x \cdot y \leqslant z \quad \Longleftrightarrow \quad x \leqslant z/y \quad \Longleftrightarrow \quad y \leqslant x\backslash z$$

holds for all $x, y, z \in A$.

As usual, we abbreviate $x \cdot y$ by xy and adopt the convention that $\cdot$ binds stronger than $/, \backslash$.

If the poset is a lattice $(A, \wedge, \vee)$ then $\mathbf{A} = (A, \wedge, \vee, \cdot, \mathbf{1}, /, \backslash)$ is a **residuated lattice**.

$\mathbf{A}$ is **idempotent** if $xx = x$ for all $x \in A$.

The cone of a residuated lattice

The **cone** of a residuated lattice $\mathbf{A}$ is the union of its **positive cone** $\uparrow\mathbf{1}$ and **negative cone** $\downarrow\mathbf{1}$.

$\mathbf{A}$ is **conical** if $A = \uparrow\mathbf{1} \cup \downarrow\mathbf{1}$.

Lemma (Stanovsky 2007)

For any ***idempotent*** *residuated lattice* $\mathbf{A}$ *and* $x, y \in A$*:*

- $x \wedge y \leqslant xy \leqslant x \vee y$.
- *If* $\mathbf{1} \leqslant xy$, *then* $xy = x \vee y$.
- *If* $xy \leqslant \mathbf{1}$, *then* $xy = x \wedge y$.
- *If* $x, y \in \uparrow\mathbf{1}$, *then* $xy = x \vee y$, *and, if* $x, y \in \downarrow\mathbf{1}$, *then* $xy = x \wedge y$.
- $\langle \downarrow\mathbf{1}, \wedge, \vee, \Rightarrow, \mathbf{1} \rangle$ *is a Brouwerian algebra, where* $x \Rightarrow y := (x\backslash y) \wedge \mathbf{1}$.

The monoidal preorder

Define the **monoidal preorder** on an idempotent residuated lattice $\mathbf{A}$ by

$$x \sqsubseteq y \iff xy = x.$$

Lemma

For any idempotent residuated lattice $\mathbf{A}$, the relation $\sqsubseteq$ is a preorder on A with top element $\mathbf{1}$, where if $\mathbf{A}$ has a least element, this is also the least element of $\sqsubseteq$. Moreover, for any $x, y \in A$:

- *If $\mathbf{1} \leqslant x, y$, then $x \leqslant y \iff y \sqsubseteq x$.*
- *If $x, y \leqslant \mathbf{1}$, then $x \leqslant y \iff x \sqsubseteq y$.*

Conservative residuated lattices

A residuated lattice **A** is **conservative** if $xy \in \{x, y\}$ for all $x, y \in A$.

In semigroup theory **conservative** is also called **quasitrivial**

Every conservative residuated lattice is **idempotent**.

For a conservative residuated lattice the monoidal preorder determines xy via

$$xy = \begin{cases} x & \text{if } x \sqsubseteq y \\ y & \text{otherwise.} \end{cases}$$

Lemma

*Every idempotent residuated **chain** $\mathbf{A}$ is conservative.*

Proof.

Since $\mathbf{A}$ is a chain, $\mathbf{1} \leqslant xy$ or $xy \leqslant \mathbf{1}$, hence $xy = x \vee y$ or $xy = x \wedge y$. In a total order it follows that $xy \in \{x, y\}$. □

The converse only holds for the elements in the cone.

Lemma

If $\mathbf{A}$ is a conservative residuated lattice, then $\langle {\downarrow}\mathbf{1} \cup {\uparrow}\mathbf{1}, \leqslant \rangle$ is a chain.

Proof.

$xy = x \wedge y$ for any x, y in the negative cone of $\mathbf{A}$, and therefore conservativity implies that $x \wedge y = x$ or $x \wedge y = y$, so $\langle {\downarrow}\mathbf{1}, \leqslant \rangle$ is a chain. The argument for the positive cone is symmetrical. □

Odd Sugihara monoids

A residuated lattice is commutative if $xy = yx$.

In this case $x\backslash y = y/x$ and is written as $x \to y$.

Example

The variety OSM of **odd Sugihara monoids** consists of all semilinear commutative idempotent residuated lattices satisfying $(x \to \mathbf{1}) \to \mathbf{1} \approx x$, and is generated as a quasivariety [M. Dunn 1970] by the algebra

$$\mathbf{Z} = \langle \mathbb{Z}, \wedge, \vee, \cdot, \to, 0 \rangle,$$

where $\mathbf{1} = 0$ and $\cdot$ is the meet operation of the total order

$$\ldots \prec -3 \prec 3 \prec -2 \prec 2 \prec -1 \prec 1 \prec 0,$$

A variety is called

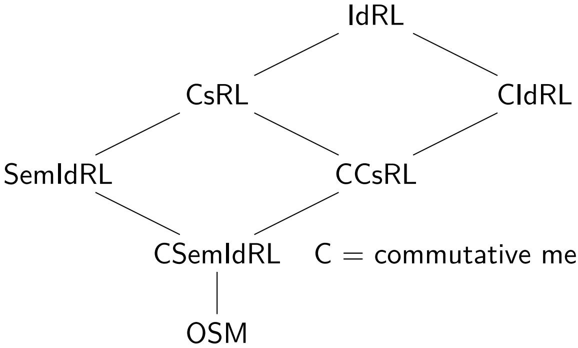
Some subvarieties of idempotent residuated lattices

IdRL = the variety of idempotent residuated lattices

CsRL = varieties generated by conservative residuated lattices

SemIdRL = varieties generated by idempotent residuated chains

OSM = variety of odd Sugihara monoids



C = commutative members

Properties of idempotent residuated lattices

LF: locally finite

FEP: finite embeddability property

AP: amalgamation property

?: is still open

Variety	LF	FEP	AP
IdRL	no	?	?
CIdRL	no	yes[vanAlten'05]	yes
CsRL	no[new Thm(1)]	yes[new Cor(2)]	?
CCsRL	?	yes[new Cor(2)]	?
SemIdRL	no[new Thm(1)]	yes[new Cor(2)]	?
SemCIdRL	yes[Raftery'07]	yes[Raftery'07]	yes[new Thm(3)]
OSM	yes[Dunn'70]	yes[Dunn'70]	yes[MarchioniMetcalfe'12]

Let $\mathbf{C} = \langle C, \leqslant \rangle$ be any chain. We say that a total order $\sqsubseteq$ on C with top element $\mathbf{1}$ is **compatible** with $\mathbf{C}$ if

1. whenever $\mathbf{C}$ has a least element $\bot$, also $\langle C, \sqsubseteq \rangle$ has least element $\bot$,
2. for all $x, y \in C$, if $\mathbf{1} \leqslant x, y$, then $x \leqslant y \iff y \sqsubseteq x$,
3. for all $x, y \in C$, if $x, y \leqslant \mathbf{1}$, then $x \leqslant y \iff x \sqsubseteq y$.

Theorem

The monoidal order $\sqsubseteq$ of any commutative idempotent residuated chain $\mathbf{A}$ is total and compatible with $\langle A, \leqslant \rangle$.

Theorem

For any chain $\mathbf{C} = \langle C, \leqslant \rangle$ *and compatible total order* $\sqsubseteq$ *on* $\mathbf{C}$*, the algebra* $\langle C, \wedge, \vee, \cdot, \mathbf{1} \rangle$ *is a commutative idempotent totally ordered monoid, where*

$$x \cdot y = \begin{cases} x & \text{if } x \sqsubseteq y, \\ y & \text{otherwise.} \end{cases}$$

Moreover, if $\mathbf{C}$ *is finite, then* $\cdot$ *has a (uniquely determined) residual* $\to$ *and* $\langle C, \wedge, \vee, \cdot, \to, \mathbf{1} \rangle$ *is a commutative idempotent residuated chain.*

[Raftery 2007] describes the structure of all finite commutative idempotent residuated chains by dividing the negative cone of such an algebra into a family of (possibly empty) intervals indexed by the positive elements.

We give a more symmetric version of this result covering **all** chains, where both negative and positive cones are divided into families of nonempty intervals with greatest elements that together form a retract of the algebra.

For any residuated lattice $\mathbf{A}$ and $a \in A$, the map $\gamma_a \colon A \to A$ mapping x to $(a/x)\backslash a$ is a closure operator on $\langle A, \leqslant \rangle$ satisfying $y \cdot \gamma_a(x) \leqslant \gamma_a(y \cdot x)$.

When $\mathbf{A}$ is commutative, the map γ_a is a nucleus on $\langle A, \leqslant \rangle$.

The algebra $\mathbf{A}_{\gamma_a} = \langle A_{\gamma_a}, \wedge, \vee_{\gamma_a}, \cdot_{\gamma_a}, \to, \gamma_a(\mathbf{1}) \rangle$ with $A_{\gamma_a} = \{\gamma_a(b) : b \in A\}$, $b \vee_{\gamma_a} c = \gamma_a(b \vee c)$, and $b \cdot_{\gamma_a} c = \gamma_a(bc)$ is always a commutative residuated lattice.

For convenience, we define ${\sim}x = x \to \mathbf{1}$.

Lemma

If $\mathbf{A}$ *is a commutative idempotent residuated chain, then* $\mathbf{A}_{\gamma_1}$ *is a retract of* $\mathbf{A}$*. Moreover, any homomorphism* $f : \mathbf{A} \to \mathbf{B}$ *between commutative idempotent residuated chains restricts to a homomorphism* $f{\restriction}A_{\gamma_1} : \mathbf{A}_{\gamma_1} \to \mathbf{B}_{\gamma_1}$*.*

Proposition

For any commutative idempotent residuated chain $\mathbf{A}$*:*

1. $\mathbf{A}_{\gamma_1}$ *is a totally ordered odd Sugihara monoid.*
2. *For each* $c \in A_{\gamma_1}$*, the set* $A_c = \{x \in A : \gamma_{\mathbf{1}}(x) = c\}$ *is an interval of* $\mathbf{A}$ *with greatest element* c*.*
3. *For all* $x, y \in A$*,*
   1. *If* $x, y \in A_c$ *for some* $c \in A_{\gamma_1}$ *with* $c \leqslant \mathbf{1}$*, then* $xy = x \wedge y$*.*
   2. *If* $x, y \in A_c$ *for some* $c \in A_{\gamma_1}$ *with* $\mathbf{1} < c$*, then* $xy = x \vee y$*.*
   3. *If* $x \in A_c, y \in A_d$ *for some* $c \neq d \in A_{\gamma_1}$*, then* $xy = x \iff cd = c$*.*
4. *For all* $x, y \in A$ *with* $x \in A_c$ *for some* $c \in A_{\gamma_1}$*,*

$$x \to y = \begin{cases} {\sim}c \vee y & \text{if } x \leqslant y, \\ {\sim}c \wedge y & \text{if } y < x. \end{cases}$$

Let **S** be any totally ordered odd Sugihara monoid and let $\mathcal{X} = \{\langle X_c, \leqslant_c \rangle : c \in S\}$ be a family of (disjoint) chains such that each $c \in S$ is the greatest element of X_c. We define for all $a, b \in S$ with $x \in X_a$ and $y \in X_b$,

$$x \preceq y \quad :\Longleftrightarrow \quad a < b \quad \text{or} \quad (a = b \ \text{and} \ x \leqslant_a y).$$

Then $\preceq$ is a total order on

$$S \otimes \mathcal{X} := \bigcup \{X_c : c \in S\}.$$

We let $\wedge$ and $\vee$ be the meet and join operations for $\preceq$ and define the algebra

$$\mathbf{S} \otimes \mathcal{X} := \langle S \otimes \mathcal{X}, \wedge, \vee, \cdot, \to, \mathbf{1} \rangle,$$

where for $a, b \in S$ and $x \in X_a, y \in X_b$,

$$x \cdot y = \begin{cases} x \wedge y & \text{if } a = b \leqslant \mathbf{1} \\ x \vee y & \text{if } \mathbf{1} < a = b \\ x & \text{if } a \neq b \text{ and } ab = a \\ y & \text{if } a \neq b \text{ and } ab = b \end{cases} \quad \text{and} \quad x \to y = \begin{cases} {\sim} a \vee y & \text{if } x \leqslant y, \\ {\sim} a \wedge y & \text{if } y < x. \end{cases}$$

Theorem

Let $\mathbf{S}$ *be any totally ordered odd Sugihara monoid and let* $\mathcal{X} = \{\langle X_c, \leqslant_c \rangle : c \in S\}$ *be a family of (disjoint) chains such that each* $c \in S$ *is the greatest element of* X_c.
Then $\mathbf{S} \otimes \mathcal{X}$ *is a commutative idempotent residuated chain satisfying* $\mathbf{S} = (\mathbf{S} \otimes \mathcal{X})_{\gamma_1}$ *and* $(S \otimes \mathcal{X})_c = X_c$ *for each* $c \in S$.
Moreover, every commutative idempotent residuated chain has this form.

Corollary (Raftery 2007)

The variety of semilinear commutative idempotent residuated lattices is locally finite.

Idempotent Residuated Chains

Since the monoidal preorder of such an idempotent residuated chain $\mathbf{A}$ is not a partial order in general, we define for $x, y \in A$,

$$x \sqsupseteq\!\!\!\sqsubseteq y \iff x \sqsubseteq y \text{ and } y \sqsubseteq x \quad \text{and} \quad x \parallel y \iff x \not\sqsubseteq y \text{ and } y \not\sqsubseteq x.$$

$x \in A$ is **central** if it commutes with every other element of $\mathbf{A}$, i.e., $xy = yx$ for all $y \in A$.

Lemma

For any idempotent residuated chain $\mathbf{A}$, if two elements do not commute, then one is positive and the other negative. Moreover, for each $x \in A$, there are two distinct possibilities:

1. *x is central and for all $y \in A$, either $x \sqsubseteq y$ or $y \sqsubseteq x$, and $x \sqsupseteq\!\!\!\sqsubseteq y \iff x = y$.*
2. *x is not central, and there is a unique $y \in A$ such that x and y do not commute.*

Idempotent Residuated Chains

For any element a of an idempotent residuated chain $\mathbf{A}$, define $a^\sharp = a$, if a is central, and otherwise, the only element of $\mathbf{A}$ that does not commute with a.

Notice that in both cases $(a^\sharp)^\sharp = a$.

If a is not central, we call $\{a, a^\sharp\}$ a **noncommuting pair**.

Lemma

For any idempotent residuated chain $\mathbf{A}$, $a \in A$, and $x \in A \backslash \{a, a^\sharp\}$,

$$a \sqsubseteq x \iff a^\sharp \sqsubseteq x \quad \textit{and} \quad x \sqsubseteq a \iff x \sqsubseteq a^\sharp.$$

We now identify properties of the monoidal preorder, analogously to the commutative case, and show that in the finite setting these properties provide a complete description of the algebra.

Laced preorders

A preorder $\sqsubseteq$ on a set A is **laced** if

1. it has a (unique) top element $\mathbf{1}$,
2. each $a \in A$ is either comparable with all the other elements and we fix $a^\sharp = a$, or there is a unique element $a^\sharp$ such that $a \sqsupset a^\sharp$ or $a \parallel a^\sharp$,
3. for all $a \in A$ and $x \in A \backslash \{a, a^\sharp\}$,

$$a \sqsubseteq x \iff a^\sharp \sqsubseteq x \quad \text{and} \quad x \sqsubseteq a \iff x \sqsubseteq a^\sharp.$$

Now let $\mathbf{C} = \langle C, \leqslant \rangle$ be any chain. We say that a laced preorder $\sqsubseteq$ on C is **compatible** with $\mathbf{C}$ if

1. any least element of $\mathbf{C}$ is also the least element of $\sqsubseteq$,
2. for all $x, y \in C$, if $\mathbf{1} \leqslant x, y$, then $x \leqslant y \iff y \sqsubseteq x$,
3. for all $x, y \in C$, if $x, y \leqslant \mathbf{1}$, then $x \leqslant y \iff x \sqsubseteq y$,
4. for each $x \in C$, if $x \neq x^\sharp$, then $\mathbf{1} \leqslant x \iff x^\sharp \leqslant \mathbf{1}$.

Theorem

1. *The monoidal preorder* $\sqsubseteq$ *of any idempotent residuated chain* $\mathbf{A}$ *is laced and compatible with* $\langle A, \leqslant\rangle$*.*
2. *For any chain* $\mathbf{C} = \langle C, \wedge, \vee\rangle$ *and compatible laced preorder* $\sqsubseteq$ *on* $\mathbf{C}$*, the algebra* $\langle C, \wedge, \vee, \cdot, \mathbf{1}\rangle$ *is an idempotent totally ordered monoid, where*

$$x \cdot y = \begin{cases} x & \text{if } x \sqsubseteq y, \\ y & \text{otherwise,} \end{cases}$$

   Moreover, if $\mathbf{C}$ *is finite, then* $\cdot$ *has (uniquely determined) residuals* $\backslash$ *and* $/$ *and* $\langle C, \wedge, \vee, \cdot, \backslash, /, \mathbf{1}\rangle$ *is an idempotent residuated chain.*

We use this representation theorem to count the number of idempotent residuated chains of size $n \geqslant 2$ up to isomorphism.

Theorem

The number $\mathbf{I}(n)$ *of idempotent residuated chains of size* $n \geqslant 2$ *is satisfies the recurrence formula*
$\mathbf{I}(2) = 1,\ \ \mathbf{I}(3) = 2,\ \ \mathbf{I}(n+2) = 2\mathbf{I}(n) + 2\mathbf{I}(n+1),$
hence the number of idempotent residuated chains of size $n \geqslant 2$ *is*

$$\mathbf{I}(n) = \frac{\left(1+\sqrt{3}\,\right)^n - \left(1-\sqrt{3}\,\right)^n}{2\sqrt{3}}.$$

Recall that the variety of semilinear **commutative** idempotent residuated lattices is locally finite.

This property fails, if semilinearity is weakened to distributivity or idempotence is weakened to being square-increasing, square-decreasing, or n-potent for $n \geqslant 3$ [Raftery 2007].

Theorem (1)

The variety of semilinear idempotent residuated lattices is not locally finite.

Proof.

It suffices to exhibit an infinite idempotent residuated chain with a finite set of generators. Consider the set $\mathbb{Z}$ of integers with the standard order, and define $x \cdot y = x$ if $|x| \geqslant |y|$ and $x \cdot y = y$ otherwise. It is easy to see that this determines the unique structure of an idempotent residuated chain. Moreover, $x \backslash x = |x|$ for each $x \in \mathbb{Z}$, and if $x > 0$, then $x \backslash 0 = -x - 1$. So we have $1 = |-1|$, $-2 = 1\backslash 0$, $2 = |-2|$, $-3 = 2\backslash 0$, etc. Also, $0 = (-1)/(-1)$. Hence $\{-1\}$ generates the whole algebra. □

Conservative Residuated Lattices

Recall that idempotent residuated lattices are **conservative** if $(\forall x)(\forall y)(xy \approx x \text{ or } xy \approx y)$.

A variety V has the **finite embeddability property** if any finite partial subalgebra of a member of V embeds into a finite member of V.

Theorem

Let K *be a class of conservative residuated lattices defined relative to* IdRL *by positive universal formulas in the language* $\{\vee, \cdot, \mathbf{1}\}$. *Then the variety* V *generated by* K *has the finite embeddability property.*

Corollary (2)

CsRL, CCsRL, *and* SemIdRL *have the finite embeddability property.*

Catalan sums

Any finite commutative conservative residuated lattice is subdirectly irreducible, since its negative cone is a finite chain of (central) idempotents.

The class $\mathsf{Si}(\mathsf{CCsRL})_{fin}$ of finite subdirectly irreducible members of CCsRL therefore consists of all finite commutative conservative residuated lattices and generates CCsRL.

Moreover, using B. McCune's Mace4, it can be shown that there are 1, 2, 5, 14, 42, 132, 429, 1430, 4862, and 16796 such algebras of size 2 to 11, respectively.

This sequence corresponds exactly to the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Catalan sums

For $\mathbf{A}, \mathbf{B} \in \mathrm{Si}(\mathrm{CCsRL})_{fin}$, define the **Catalan sum** $\mathbf{C} = \mathbf{A} \circledcirc \mathbf{B}$ as follows: Let C be the disjoint union of A and B and define a lattice order

$$\leqslant^{\mathbf{C}} = \leqslant^{\mathbf{A}} \cup \leqslant^{\mathbf{B}} \cup (\{\bot^{\mathbf{A}}\} \times B) \cup (A \times \uparrow \mathbf{1}^{\mathbf{B}}).$$

To specify $\cdot^{\mathbf{C}}$, it suffices to define the following monoidal order:

$$\sqsubseteq^{\mathbf{C}} = \sqsubseteq^{\mathbf{A}} \cup \sqsubseteq^{\mathbf{B}} \cup (\{\bot^{\mathbf{A}}\} \times B) \cup (B \times (A \backslash \{\bot^{\mathbf{A}}\})).$$

Informally, the monoidal order of $\mathbf{C}$ is the ordinal sum $\{\bot^{\mathbf{A}}\} \oplus \langle B, \sqsubseteq \rangle \oplus \langle A \backslash \{\bot^{\mathbf{A}}\}, \sqsubseteq \rangle$. Since the top element is always the identity, it follows that if $\mathbf{A}$ is nontrivial, then $\mathbf{1}^{\mathbf{C}} = \mathbf{1}^{\mathbf{A}}$ and otherwise $\mathbf{1}^{\mathbf{C}} = \mathbf{1}^{\mathbf{B}}$.

The lattice order implies that $\bot^{\mathbf{A}}$ is the least element of $\mathbf{C}$, and that $a \vee b = \mathbf{1}^{\mathbf{B}} \vee b$ whenever $a \in A \backslash \{\bot^{\mathbf{A}}\}$ and $b \in B$.

If $\mathbf{A}$ or $\mathbf{B}$ is a one element algebra, then the underlying lattice of $\mathbf{C}$ is simply the ordinal sum of the lattices of $\mathbf{A}$ and $\mathbf{B}$.

Catalan sums

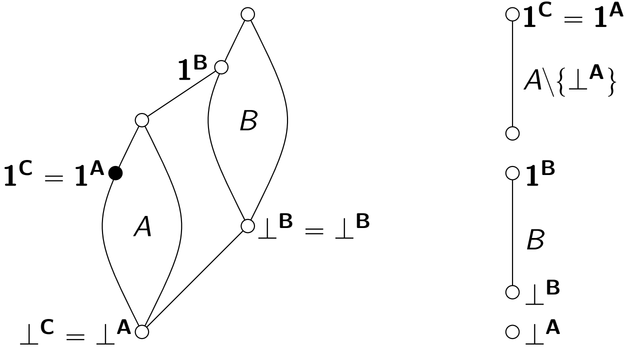


Figure: The Catalan sum $\mathbf{C} = \mathbf{A} \circledc \mathbf{B}$, for nontrivial $\mathbf{A}$.

Catalan algebras

Lemma

If $\mathbf{A}, \mathbf{B} \in \mathrm{Si}(\mathrm{CCsRL})_{fin}$*, then* $\mathbf{A} \copyright \mathbf{B} \in \mathrm{Si}(\mathrm{CCsRL})_{fin}$*.*

Lemma

Suppose that $\mathbf{C} \in \mathrm{Si}(\mathrm{CCsRL})_{fin}$ *has size* $n \geqslant 2$*. Then* $\mathbf{C} = \mathbf{A} \copyright \mathbf{B}$ *for a pair* $\mathbf{A}, \mathbf{B} \in \mathrm{Si}(\mathrm{CCsRL})_{fin}$ *that is unique up to isomorphism.*

Catalan algebras

An algebra is **Catalan** if it is a one-element residuated lattice or a Catalan sum of Catalan algebras.

In particular, if $\mathbf{C}_1^1$ is a one-element residuated lattice, then $\mathbf{C}_1^2 = \mathbf{C}_1^1 \copyright \mathbf{C}_1^1$ is the two element Boolean algebra.

The two three element chains are $\mathbf{C}_1^3 = \mathbf{C}_1^1 \copyright \mathbf{C}_1^2$ and $\mathbf{C}_2^3 = \mathbf{C}_1^2 \copyright \mathbf{C}_1^1$.

In general, the algebras of size n are built by constructing all Catalan sums of algebras $\mathbf{A}$ and $\mathbf{B}$ of size $n - k$ and k respectively, as k ranges from 1 to $n - 1$.

Theorem

The class of finite conservative commutative residuated lattices is precisely the class of Catalan algebras.

Catalan algebras

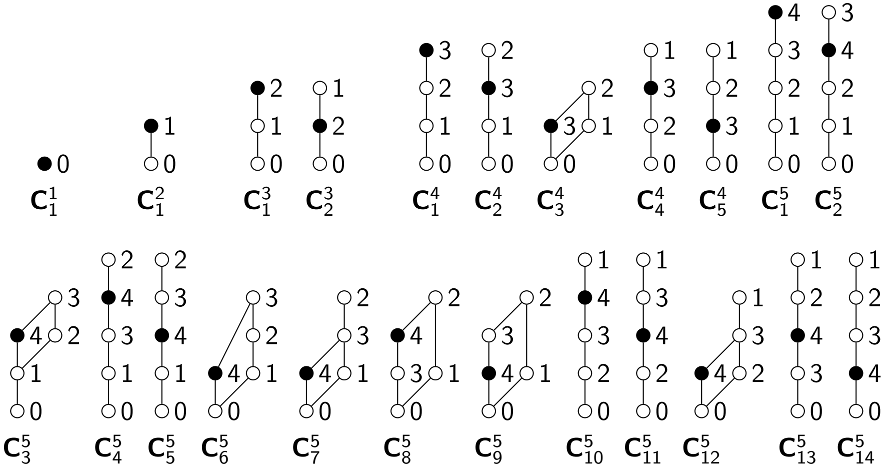


Figure: Subdirectly irred. commutative conservative residuated lattices of size $\leqslant 5$

This yields the following result.

Theorem

The number of conservative commutative residuated lattices of $n \geqslant 1$ elements is $\mathbf{C}(n) = \frac{1}{n}\binom{2(n-1)}{n-1}$, that is, the $(n-1)$th Catalan number.

Proof.

We will prove the result by induction. The sequence $(C_i : i \geqslant 0)$ of Catalan numbers is determined by $C_0 = 1$ and $C_{n+1} = \sum_{i=1}^{n} C_i C_{n-i}$. Obviously, $\mathbf{C}(1) = 1 = C_0$. Suppose now that $n > 1$. Then

$$\begin{aligned} \mathbf{C}(n+1) &= \sum_{k=1}^{n} \mathbf{C}(k) \cdot \mathbf{C}(n+1-k) \\ &= \sum_{k=1}^{n} C_{k-1} C_{n-k} = \sum_{i=0}^{n-1} C_i C_{n-1-i} = C_{n-1}. \end{aligned}$$

□

The Amalgamation Property

A **span** of a class of algebras K is a pair of embeddings $\langle i_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}, i_2 \colon \mathbf{A} \hookrightarrow \mathbf{C} \rangle$ between algebras $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$.

K has the **amalgamation property** if for every span of K, there exist $\mathbf{D} \in \mathsf{K}$ and embeddings $j_1 \colon \mathbf{B} \hookrightarrow \mathbf{D}$ and $j_2 \colon \mathbf{C} \hookrightarrow \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$.

The main tools for proving AP will be the characterization of commutative idempotent residuated chains and the following criterion for amalgamation in varieties of semilinear residuated lattices.

Theorem (Metcalfe, Montagna, Tsinakis 2014)

Let V *be a variety of semilinear residuated lattices with the congruence extension property, and let* $\mathcal{T}$ *be the class of finitely generated totally ordered members of* V*. If every span in* $\mathcal{T}$ *has an amalgam in* V*, then* V *has the amalgamation property.*

AP for commutative semilinear idempotent RLs

From the structural description of commutative idempotent residuated chains:

Lemma

The class of commutative idempotent residuated chains has the amalgamation property.

Since every variety of commutative residuated lattices has the congruence extension property, we get the following result.

Theorem (3)

The variety of semilinear commutative idempotent residuated lattices has the amalgamation property.

Does there exists a noncommutative variety of residuated lattices that has the amalgamation property?

There are two nonisomorphic noncommutative idempotent residuated chains of size 4: $\mathbf{C}_4$ shown below, and its "opposite" (i.e., the algebra resulting from swapping the order of the product).

Figure: The algebra $\mathbf{C}_4$.

To show that $\mathsf{V}(\mathbf{C}_4)$ has the congruence extension property, we make use of some results of [van Alten 2005].

Recall that a term $u(\vec{x}, \vec{y})$ is an **ideal term** in $\vec{x}$ for a variety V with respect to a (term-definable) constant 1 if and only if $\mathsf{V} \models t(\vec{1}, \vec{y}) \approx 1$.

The **ideals** (with respect to 1) of an algebra $\mathbf{A} \in \mathsf{V}$ are the subsets $I \subseteq A$ such that $u(\vec{a}, \vec{b}) \in I$ for every ideal term $u(\vec{x}, \vec{y})$ and $\vec{a} \in I$, $\vec{b} \in A$.

If $\mathbf{A}$ is a residuated lattice, then the ideals with respect to $\mathbf{1}$ coincide with the convex normal subalgebras of $\mathbf{A}$.

[van Alten 2005] proved that the variety generated by a residuated lattice $\mathbf{A}$ has equationally definable principal congruences (and therefore the congruence extension property) if there exists a finite set J of ideal terms (with respect to $\mathbf{1}$) such that for all $a, b \in {\downarrow}\mathbf{1}$, there exists $u(x, y) \in J$ satisfying

$$b \in \langle a \rangle^{\mathbf{A}} \iff b = u^{\mathbf{A}}(a, b),$$

where $\langle a \rangle^{\mathbf{A}}$ denotes the convex normal subalgebra generated by a.

Lemma

$V(\mathbf{C}_4)$ *has the congruence extension property.*

Proof.

Observe first that $\mathbf{C}_4$ has only the trivial proper subalgebra, since $c^\sharp = c\backslash\mathbf{1} = \bot\backslash\mathbf{1}$, $\bot = c^\sharp\backslash\mathbf{1} = \mathbf{1}/c^\sharp$, and $c = \mathbf{1}/c^\sharp$. It suffices now to check that $\mathbf{C}_4$ satisfies (35) for the set of ideal terms $J = \{\mathbf{1}, x, (x\backslash\mathbf{1})\backslash\mathbf{1}, \mathbf{1}/(\mathbf{1}/x)\}$. For $a = \mathbf{1}$, we have $\langle\mathbf{1}\rangle^{\mathbf{C}_4} = \{\mathbf{1}\} = J(\mathbf{1})$, and for $a \neq \mathbf{1}$, we have $\langle a\rangle^{\mathbf{C}_4} = C_4$ and $J(a) = \{\mathbf{1}, c, \bot\}$. □

Theorem

$V(\mathbf{C}_4)$ *has the amalgamation property.*

Involutive residuated lattices

An **involutive** residuated lattice $\langle A, \wedge, \vee, \cdot, \mathbf{1}, /, \backslash, \sim, -\rangle$ is a residuated lattice with two **linear negations** that satisfy ${\sim}x = x\backslash\mathbf{0}$, $-x = \mathbf{0}/x$, ${\sim}{-}x = x = -{\sim}x$ where $\mathbf{0} = -\mathbf{1} = {\sim}\mathbf{1}$

An involutive residuated lattice is **cyclic** if ${\sim}x = -x$.

$\cdot$ commutative $\implies$ cyclicity.

An **integral** (i.e. $\mathbf{1}$ = top element) **idempotent involutive** residuated lattice is a **Boolean algebra**.

Sugihara chains are the only commutative idempotent involutive residuated chains.

Theorem

*Every **cyclic** idempotent involutive residuated lattice is commutative.*
*Every **finite** idempotent involutive residuated **chain** is commutative.*

A noncyclic idempotent involutive residuated lattice

Let $A = \mathbb{Z} \oplus \{\mathbf{1}\} \oplus \mathbb{Z}^\partial$, where $\oplus$ is the ordinal sum.

Lattice order:

$\cdots a_{-2} < a_{-1} < a_0 < a_1 < a_2 \cdots < \mathbf{1} < \cdots b_2 < b_1 < b_0 < b_{-1} < b_{-2} \cdots$

Compatible monoid preorder:

$\cdots a_{-2} \sqsupseteq b_{-2} \sqsubset a_{-1} \sqsupseteq b_{-1} \sqsubset a_0 \sqsupseteq b_0 \sqsubset a_1 \sqsupseteq b_1 \sqsubset a_2 \sqsupseteq b_2 \sqsubset \cdots \sqsubset \mathbf{1}$

Linear negations:

$\mathbf{1} = \mathbf{0}$, $\sim a_i = b_i$, $\sim b_i = a_{i-1}$, $-a_i = b_{i+1}$, $-b_i = a_i$

A typical finite commutative idempotent involutive RL

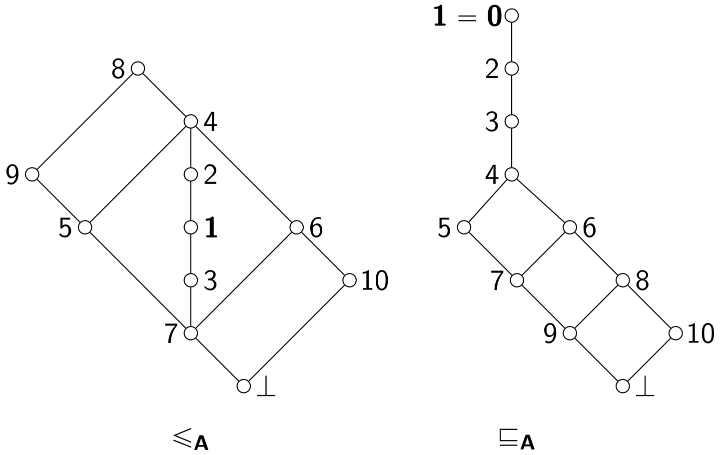


Figure: The two partial orders $\leqslant_{\mathbf{A}}$ and $\sqsubseteq_{\mathbf{A}}$ of an algebra $\mathbf{A} \in \mathsf{IdInRL}$

Commutative IdInRLs are disjoint unions of Boolean algebras

In an involutive residuated lattice, idempotence implies that $0 \leqslant 1$ and that $([0,1], \cdot, +, -, 0, 1)$ is a Boolean algebra, where $x + y = \sim(-y \cdot -x)$.

For A in CIdInRP, define the terms $0_x = -x \cdot x$ and $1_x = -(-x \cdot x)$, and let $[\![a, b]\!] = \{c \in A : ac = a, bc = c\}$.

Theorem

The semilattice intervals $([\![0_x, 1_x]\!], \cdot, +, -, 0_x, 1_x)$ are also Boolean algebras and they partition A.

We conjecture that all finite idempotent involutive residuated lattices are commutative and that the monoidal semilattice orders are distributive (checked up to size 16).

Thanks!