Generalized bunched implication algebras

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Outline

Structure of the talk

- Motivation and examples
- Congruences

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Bunched Implication Logic

- Motivated by separation logic used in pointer management in computer science.
- It is a substuctural logic and it combines an additive (Heyting) implication and a multiplicative (linear) implication.

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- $(L, \wedge, \vee)$ is a lattice,
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In general the lattice reduct need not be distributive, as in the lattice of ideals of a ring.
$I \wedge J = I \cap J$,
$I \vee J = I + J$, and
IJ contains finite sums of products ij, as usual.

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- Lattice-ordered groups
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A *Generlized Bunched Implication algebra* (or *GBI algebra*) $\mathbf{A} = (A \wedge, \vee, \cdot, \backslash, /, 1, \to, \top)$ supports two residuated structures: a residuated lattice $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ and a Brouwerian/Heyting algebra $(A, \wedge, \vee, \to, \top)$.

Relation algebras

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Given a set P for binary relations $R, S \in \mathcal{P}(P \times P)$, we define

- $R \wedge S = R \cap S$
- $R \vee S = R \cup S$
- $R \cdot S = R \circ S$ (relational composition)
- $R \to S = R^c \cup S = (R \cap S^c)^c$
- $R \backslash S = (R^\cup \circ S^c)^c$ (where $R^\cup$ is the converse of R)
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This is an example of a GBI algebra, and part of is special nature is the fact that the Heyting algebra reduct is actually Boolean. We consider generalizations of these algebras called weakening relation algebras.

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We define the set $Wk(\mathbf{P})$ of $\leq$-*weakening relations*, that is of all binary relations R on P such that $a \leq b \; R \; c \leq d$ implies $a \; R \; d$, for all $a, b, c, d \in P$.

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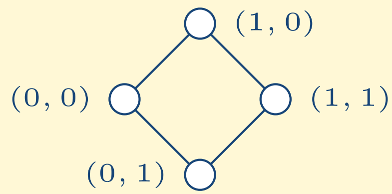
We explain why $Wk(\mathbf{P})$ supports a structure of a GBI-algebra, under union and intersection, and composition of relations.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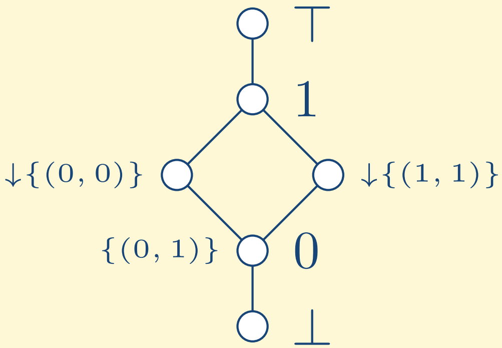
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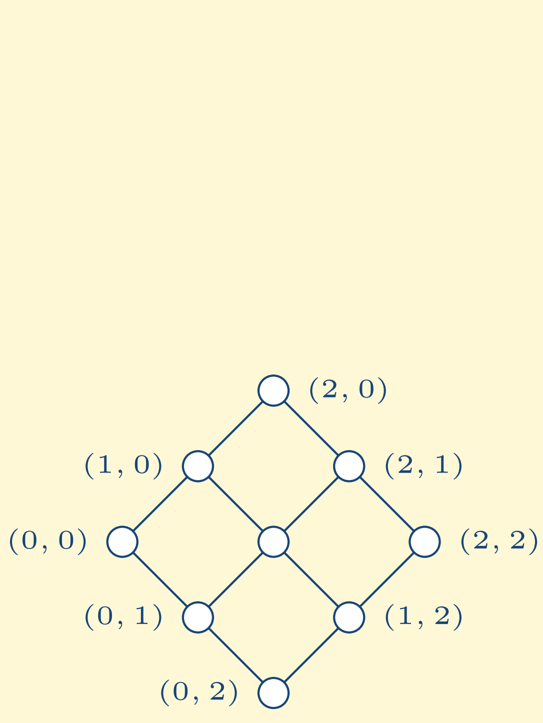
Weakening relations on 3



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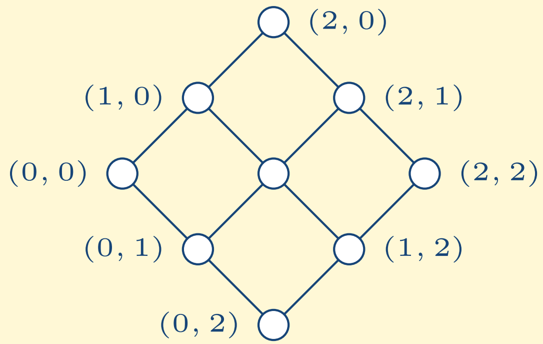

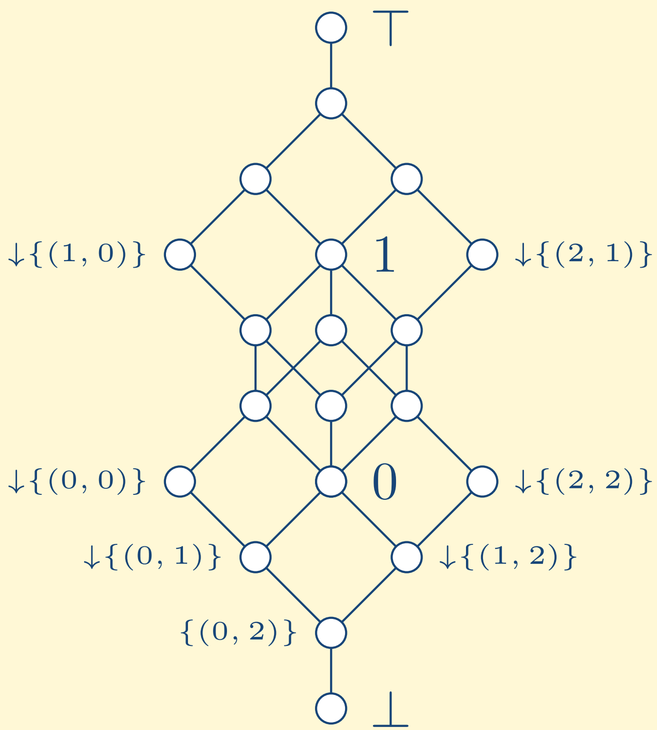

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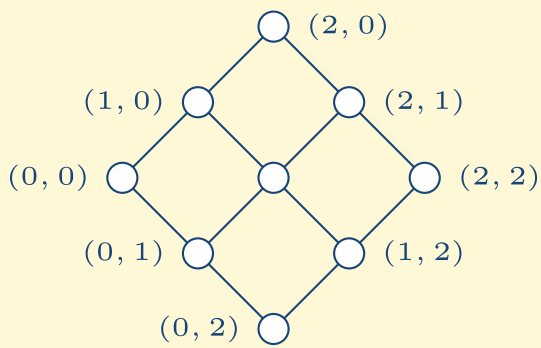

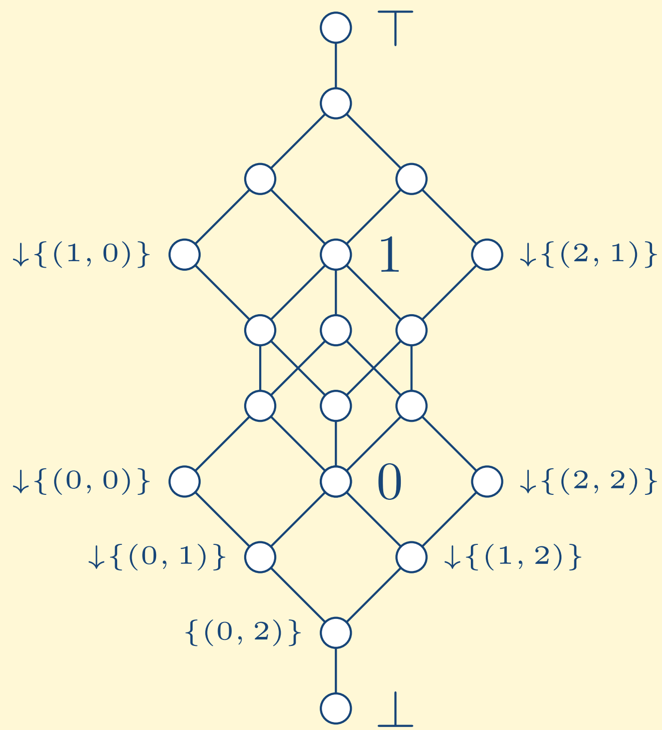

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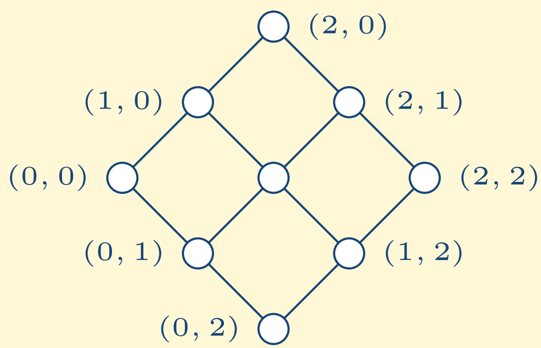

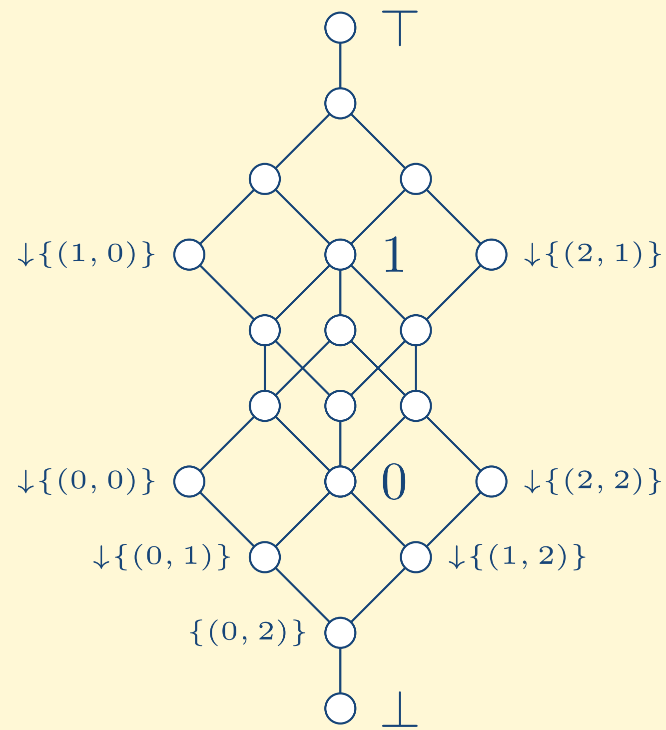

We also have that $Wk(\mathbf{P}) \cong Res(\mathcal{O}(\mathbf{P}))$. For a complete join semilattice $\mathbf{L}$, $Res(\mathbf{L})$ is the residuated lattice of all residuated maps on $\mathbf{L}$;a map on f on a poset $\mathbf{Q}$ is called *residuated* if there exists a map f^* on Q such that $f(x) \leq y$ iff $x \leq f^*(y)$, for all $x, y \in Q$.

Conuclei

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Then $\sigma[\mathbf{A}] = (\sigma[A], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma)$ is a residuated lattice-ordered semigroup, where $x \bullet_\sigma y = \sigma(x \bullet y)$, where $\bullet \in \{\wedge, \backslash, /\}$.

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Given a residuated lattice $\mathbf{A}$ and a positive idempotent element p $(1 \leq p = p^2)$, the map σ_p, where $\sigma_p(x) = p\backslash x/p$, is a topological conucleus called the *double division conucleus by p*.

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conuclei

It turns out that for all $x \in A$, $x = p\backslash x/p$ iff $x = pxp$. Even though these two maps are very different (one is an interior operator and the other is a close operator), they have the same image/fixed elements. So, $\sigma_p[A] = \{pxp : x \in A\}$.

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Recall that an *involutive* residuated lattice is an expansion of a residuated lattice with an extra constant 0 such that $\sim(-x) = x = -(\sim x)$, where $\sim x = x\backslash 0$ and $-x = 0/x$; we also define $x + y = \sim(-y \cdot -x)$.

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The involution is called *cyclic* if $\sim x = -x$ for all x.

WK as a conucleus image

Given a poset $\mathbf{P} = (P, \leq)$, we set $\mathbf{A} = Rel(P)$, to be the cyclic involutive GBI algebra of all binary relations on the set P. Note that $p = \leq$ is a positive idempotent element of $\mathbf{A}$. It is easy to see that $p \backslash \mathbf{A} / p$ is exactly $Wk(\mathbf{P})$, so the latter is a cyclic involutive GBI-algebra.

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For every relation r, $\sim r = -r$ is the converse of the complement of r. (Note that neither the complement nor the converse of a weakening relation is again a weakening relation.)

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Every lattice-ordered (pre)group can be embedded as a residuated lattice in $Wk(\mathbf{C})$, where $\mathbf{C}$ is a chain. (Lattice-ordered pregroups are involutive residuated lattices where $x + y = x \cdot y$.) The subalgebra of $Wk(\mathbf{C})$ that is the image of the embedding is also involutive, but with negation constant 1 (and for pregroups it is not cyclic).

More examples

We can take $\mathbf{A}$ to be $\mathcal{P}(\mathbf{M})$, where $\mathbf{M}$ is a monoid. The positive idempotent elements ($1 \leq p = p^2$) of $\mathcal{P}(\mathbf{M})$ are exactly the submonoids of $\mathbf{M}$.

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The elements of $p\backslash\mathcal{P}(\mathbf{M})/p$ turn out to be the downsets under a particular auxiliary order $\leq_p$: $x \leq_p y$ iff $x = ayb$ for some $a, b \in p$. Since p is positive idempotent, the relation $\leq_p$ is a preorder and p is its negative cone.

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We can take $M = \mathbb{Z}$ and $p = \mathbb{N}$. Then $\le_p$ is the usual order on $\mathbb{Z}$ and $\mathbb{N}\backslash\mathcal{P}(\mathbb{Z})/\mathbb{N}$ is isomorphic to $\mathbb{Z}$ extended with a top and a bottom element, which is an involutive GBI algebra.

More examples

As another example we can take $M = \mathbb{N}$ and $p = E$, the set of even numbers. The fixed sets of the conucleus are unions $\uparrow e \cup \uparrow o$, where $e \in \bar{E}$ and $o \in \bar{O}$.

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So, $E \backslash \mathcal{P}(\mathbb{N})/E$ is isomorpic to $\bar{E} \times \bar{O}$. The operation is given by $(e_1, o_1) + (e_2, o_2) = ((e_1 + e_2) \wedge (o_1 + o_2), (e_1 + o_2) \wedge (o_1 + e_2))$.

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$$(e, o) \equiv \begin{bmatrix} e & o \\ o & e \end{bmatrix}$$

Congruences in RL

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Alternative subsets to F include convex normal (for $\rho_a x = (ax/a) \wedge 1$ and $\lambda_a(x) = (a\backslash xa) \wedge 1$)) subalgebras, such as $\{x : \exists f \in F.\; f \leq x \leq 1/f\}$ and also convex normal negative submonoids, such as the negative cone of F: $\{x \in F : x \leq 1\}$.

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Note that if A is a Brouwerian or a Heyting algebra, then all notions coincide.

Congruences in GBI

GBI-congruences are RL-congruences with further closure conditions. As a result the upset of the equivalence class of 1 is a normal submonoid filter with further closure conditions. We identify these as closure under
$r_{a,b}(x) = (a \to b)/(xa \to b)$ and
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Alternatively, congruences are characterized by their equivalence classes of $\top$. These are usual lattice filters that are closed under
$u_{a,b}(x) = a/(b \wedge x) \to a/b$,
$u'_{a,b}(x) = (b \wedge x)\backslash a \to b\backslash a$,
$v_{a,b}(x) = ab \to (a \wedge x)b$,
$v'_{a,b}(x) = ab \to a(b \wedge x)$, and
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$w(x) = \top\backslash x/\top$, for all a, b.

As a result we obtain a parameterized local deduction theorem for the GBI.

Congruences in InGBI

The congruence class of $\top$ in involutive GBI-algebras is characterized as filters closed under the terms

$\neg{\sim}x$

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and ${\sim}(\top(-x)\top)$,

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The latter specialises to the known characterization of congruences in relation algebras as *ideals* closed under the term $\top x \top$.

FEP

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Using well quasiorders and better quasiorders we can show the FEP for many subvarieties of GBI for which multiplication distributes over meet (*fully distributive GBI algebras*). [Joint work with Riquelmi Cardona]

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For example, the FEP holds for fully distributive integral GBI-algebras. Also, for fully distributive GBI-algebras that satisfy a non-trivial equation of the form $x^n \leq x^m$ and commutativity (or various generalizations of commutativity such as $xyx = xxy$).