# The complexity of the equation solvability and equivalence problems over finite groups

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BLAST 2019

March 23, 2019

# Equivalence problem

- given: finite algebra $\mathcal{A}$
- input: $p(x_1, \ldots, x_n), q(x_1, \ldots, x_n)$ polynomials
- question:

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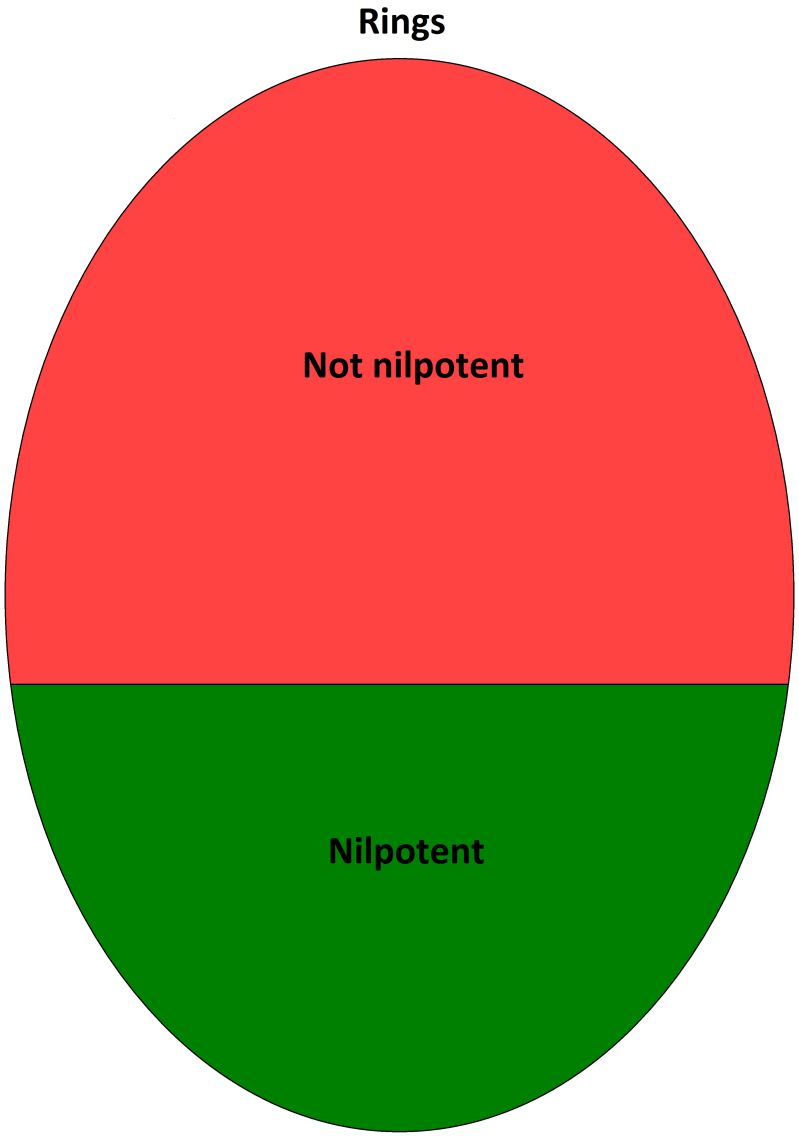
## Example

$$S, T \in \mathbf{G}[x_1, \ldots, x_n]: \qquad S \stackrel{?}{\equiv} T$$

$$S \cdot T^{-1} \stackrel{?}{\equiv} \mathrm{id}$$

*every* $g \in \mathbf{G} \setminus \{\mathrm{id}\}$: $\qquad S \cdot T^{-1} \stackrel{?}{\neq} g$

# Rings




Theorem (Burris, Lawrence (1993), Horváth (2011))

$\mathcal{R}$ *not nilpotent* $\Longrightarrow$ *NP-complete*

$\mathcal{R}$ *nilpotent* $\Longrightarrow$ *in P*

# Sigma problem

## Example

$$\underbrace{(x_1 + y_1) \cdot \ldots \cdot (x_n + y_n)}_{n \text{ factors}} = \underbrace{x_1 \cdot \ldots \cdot x_n + \cdots + y_1 \cdot \ldots \cdot y_n}_{2^n \text{ summands}}$$

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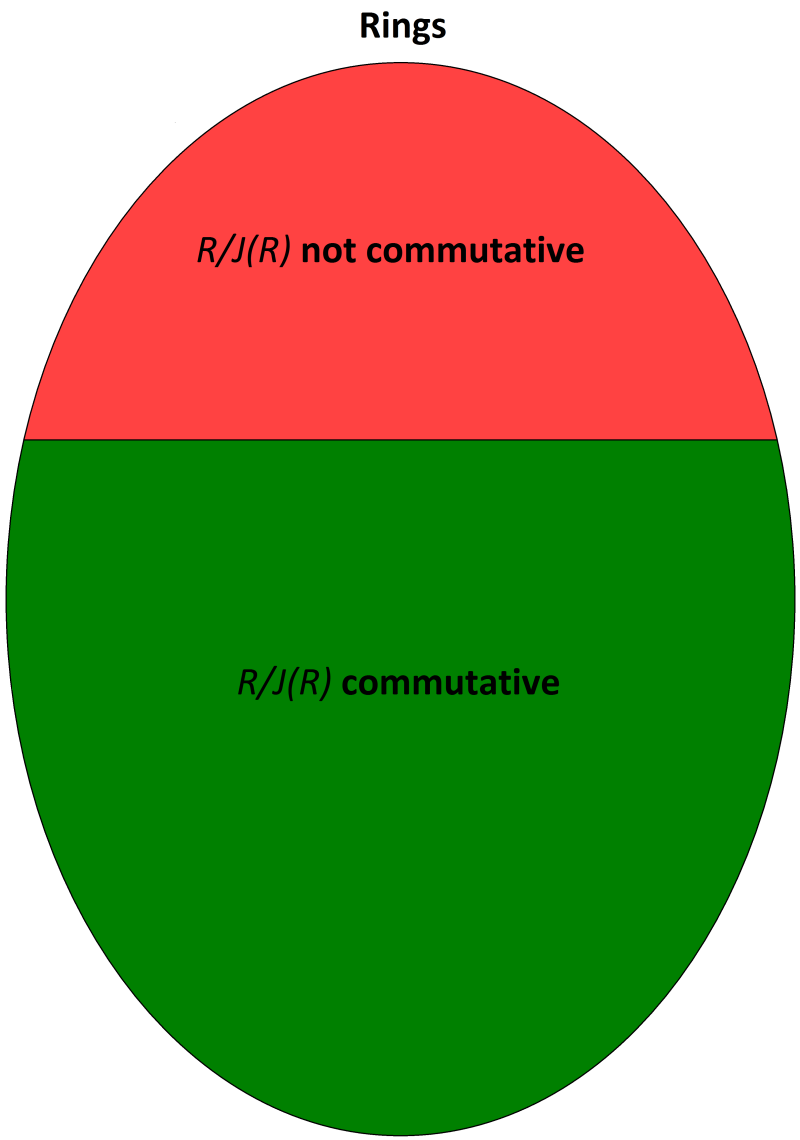
## Lawrence, Willard:

sigma equivalence, sigma solvability:
inputs are given as sums of monomials

# Rings

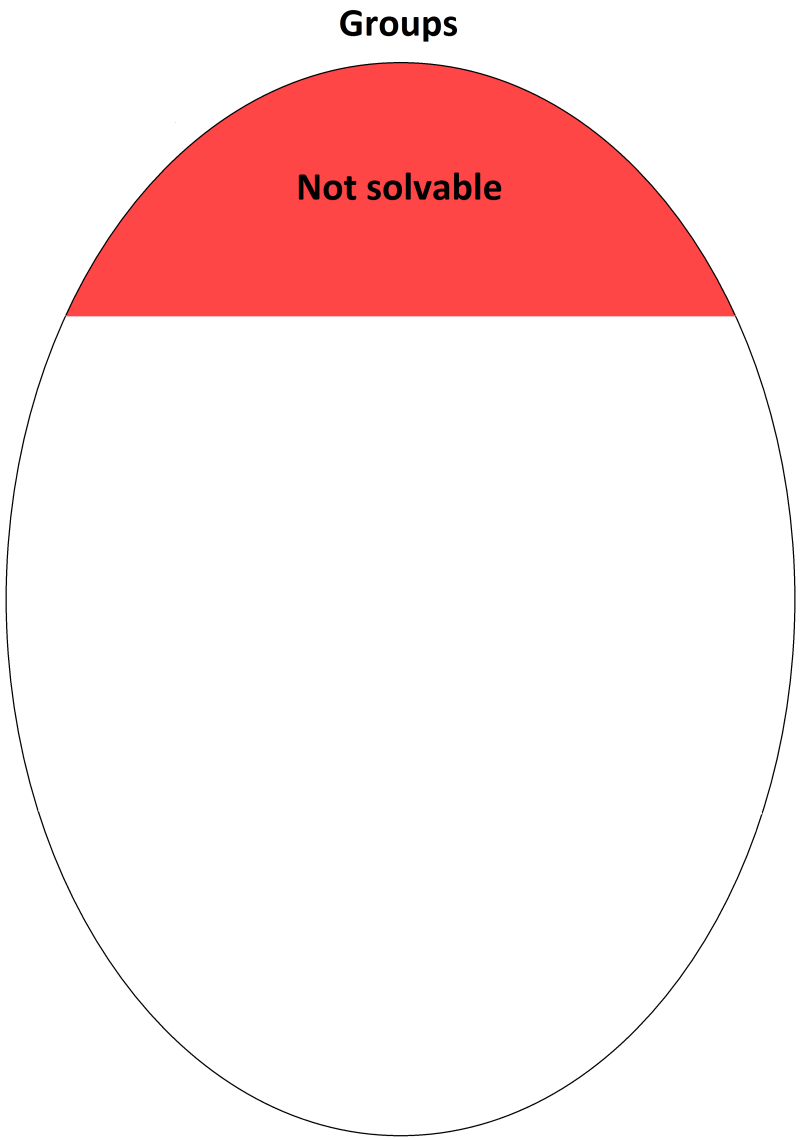



Theorem (Szabó, Vértesi (2011), Horváth, Lawrence, Willard)

$\mathcal{R}/\mathcal{J}(\mathcal{R})$ *not comm.* $\Longrightarrow$ *sigma problem is NP-complete*

$\mathcal{R}/\mathcal{J}(\mathcal{R})$ *comm.* $\Longrightarrow$ *sigma problem is in P*

# Solvability over groups

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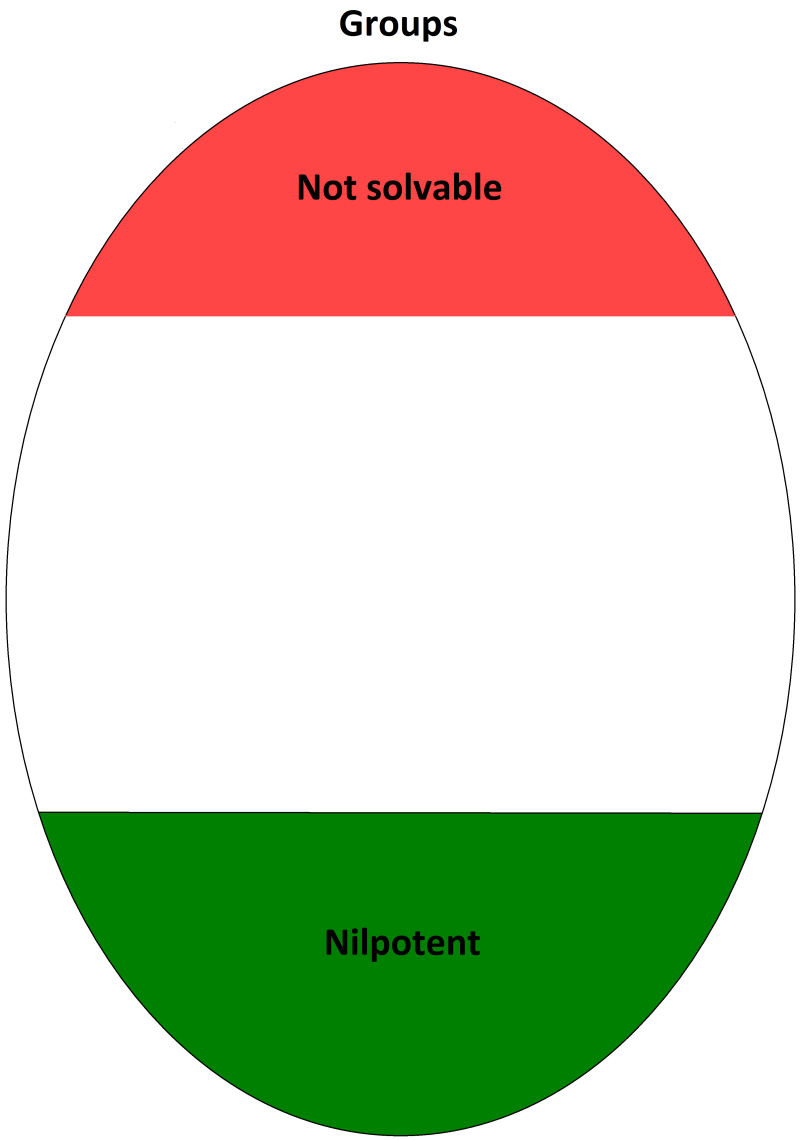


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  $Ramsey \Longrightarrow O\left(n^{|\mathbf{G}|^{\cdots^{|\mathbf{G}|}}}\right)$

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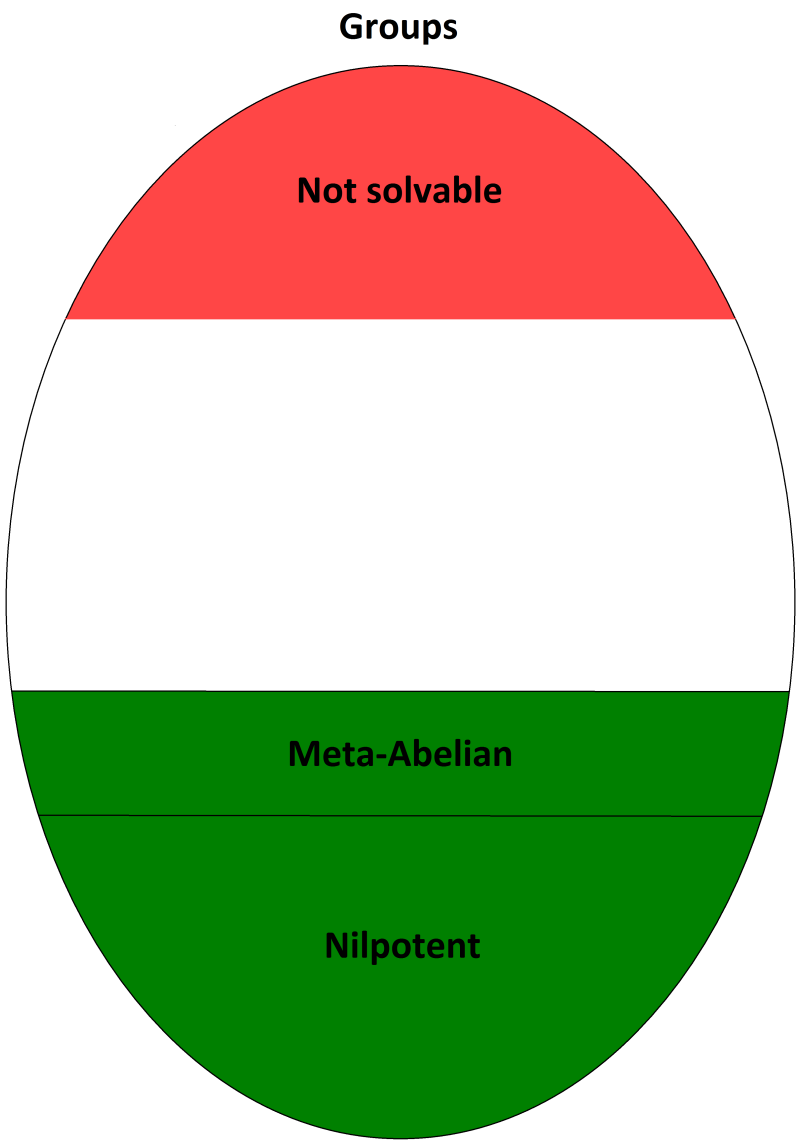


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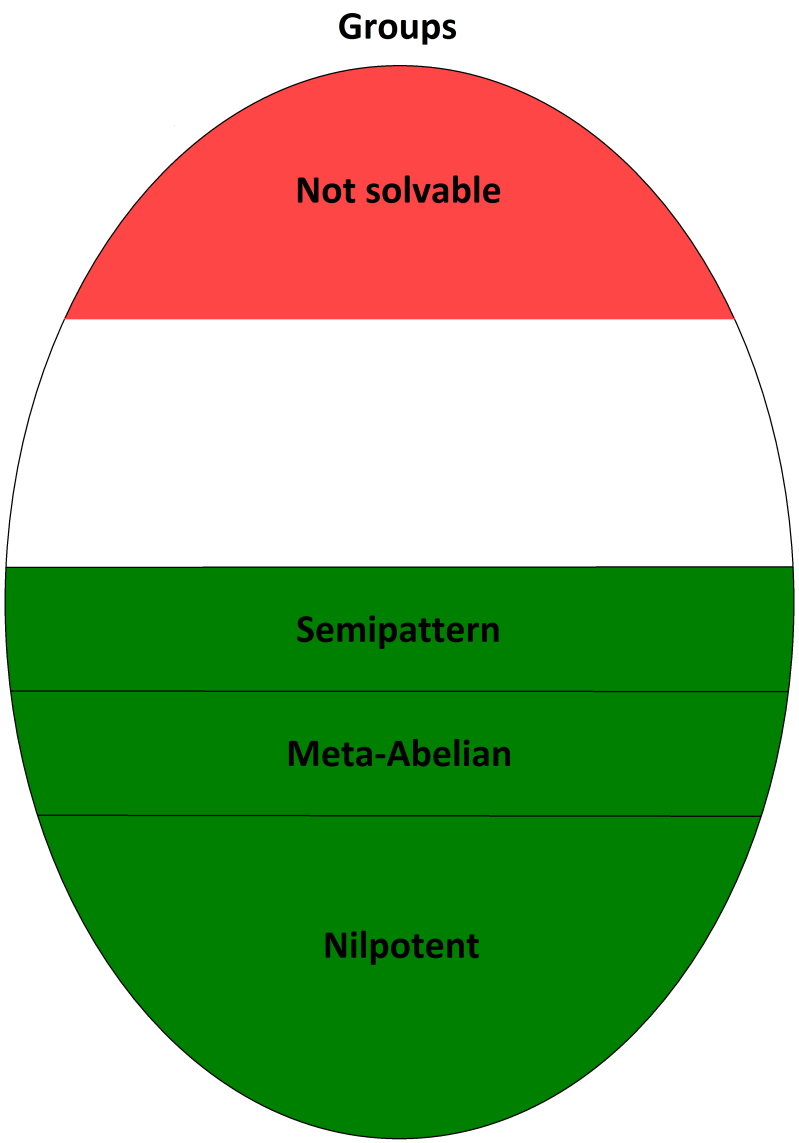


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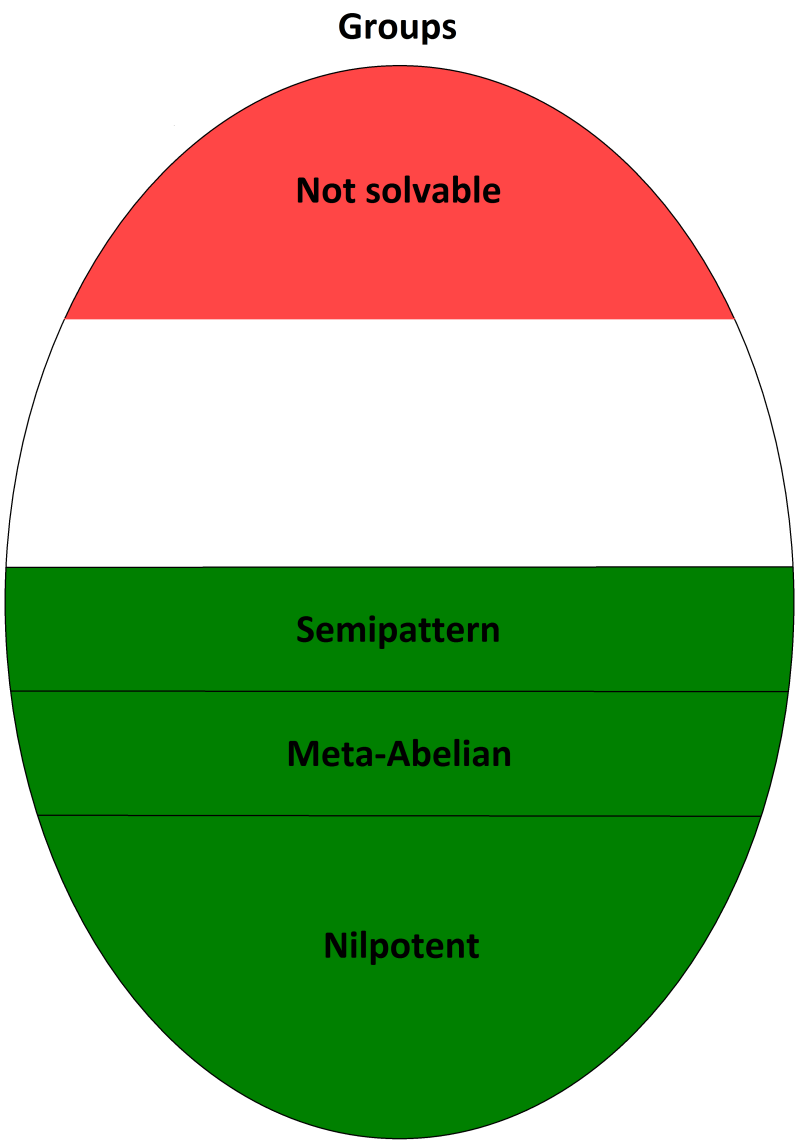


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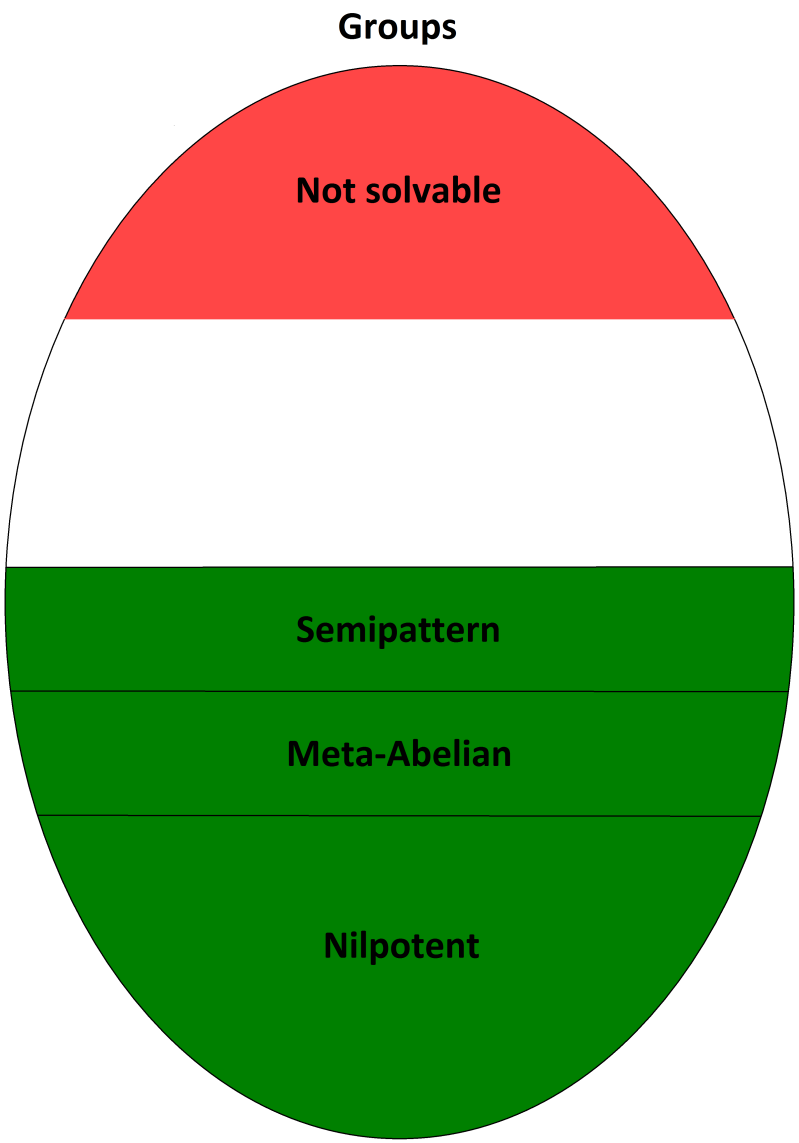


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$\mathbf{S}_4$, $\mathbf{SL}(2, \mathbb{Z}_3)$

# The group $\mathbf{SL}(2, \mathbb{Z}_3)$

$$\mathbf{SL}(2, \mathbb{Z}_3) \cong \mathbf{Q} \rtimes \mathbf{C}_3$$

$$\mathbf{Q} = \{1, -1, i, -i, j, -j, k, -k\},\ \mathbf{C}_3 = \{1, \alpha, \alpha^2\}:$$

$$i^\alpha = j \qquad j^\alpha = k \qquad k^\alpha = i$$

# Multiplication in the group $\mathbf{Q}$

## Polycyclic generating system

$u \in \mathbf{Q} \Rightarrow \exists!\, x, y, z \in \{0,1\} : u = (-1)^x \cdot i^y \cdot j^z$

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$$f_1, f_2, f_3 \in \mathbb{Z}_2[x_1, y_1, z_1, y_2, x_2, z_2]$$

$$f_1(\dots) = x_1 + x_2 + y_1 \cdot y_2 + z_1 \cdot z_2 + z_1 \cdot y_2$$

$$f_2(\dots) = y_1 + y_2$$

$$f_3(\dots) = z_1 + z_2$$

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$$F_1(\ldots) = \sum_{k=1}^{n} x_k + \sum_{l=2}^{n} \sum_{k=1}^{l-1} y_k y_l + \sum_{l=2}^{n} \sum_{k=1}^{l-1} z_k z_l + \sum_{l=2}^{n} \sum_{k=1}^{l-1} z_k y_l$$

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- conjugation:

$$\begin{array}{ll} u \in \mathbf{Q} & \implies \quad \exists!\, x,y,z \in \mathbb{Z}_2 :\ u = (-1)^x \cdot i^y \cdot j^z \\ h \in \mathbf{C}_3 & \end{array}$$

basis form:

$$\phi_1, \phi_2, \phi_3 \in \mathbb{F}_q[x,y,z,h]$$

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- semidirect product: $t_k = (u_k, h_k)$

  $T = (u_1, h_1) \cdot (u_2, h_2) \cdots (u_n, h_n) = (T_{\mathbf{Q}}, T_{\mathbf{C}_3})$

  $T_{\mathbf{Q}} = u_1 \cdot u_2^{h_1} \cdot u_3^{h_1 \cdot h_2} \cdot \ldots \cdot u_n^{h_1 \cdot h_2 \cdot \ldots \cdot h_{n-1}}$

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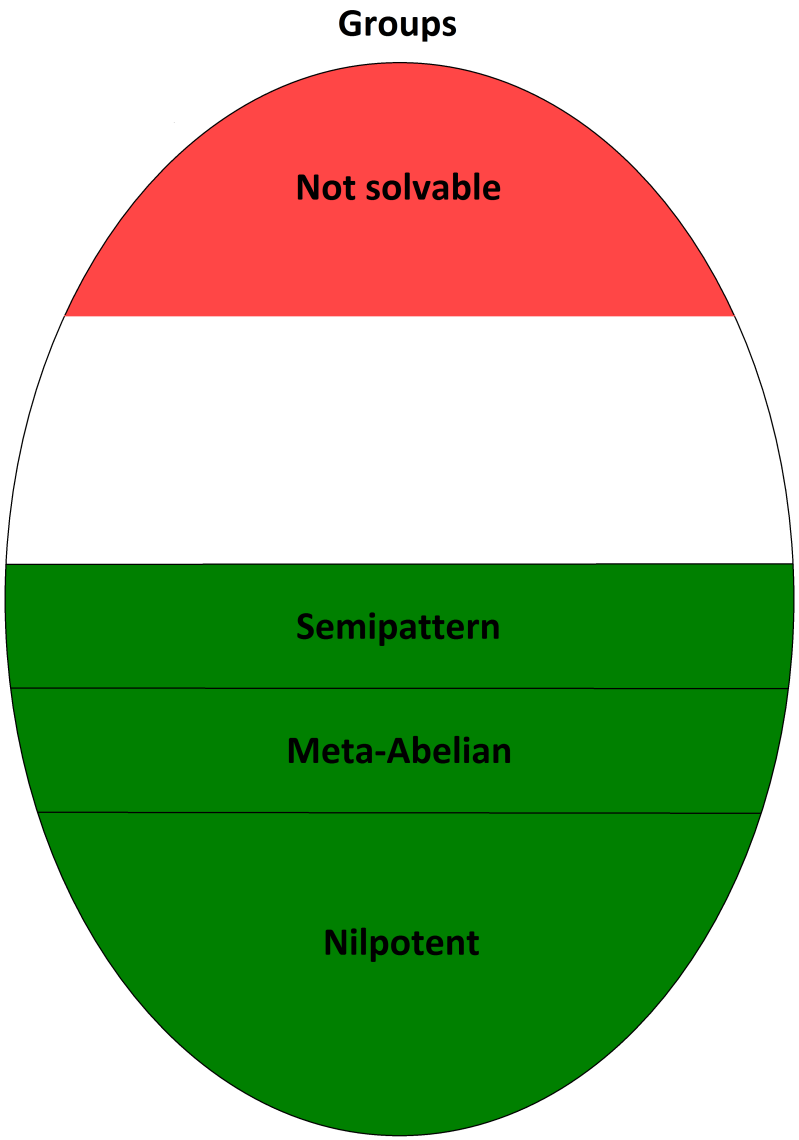
# Results



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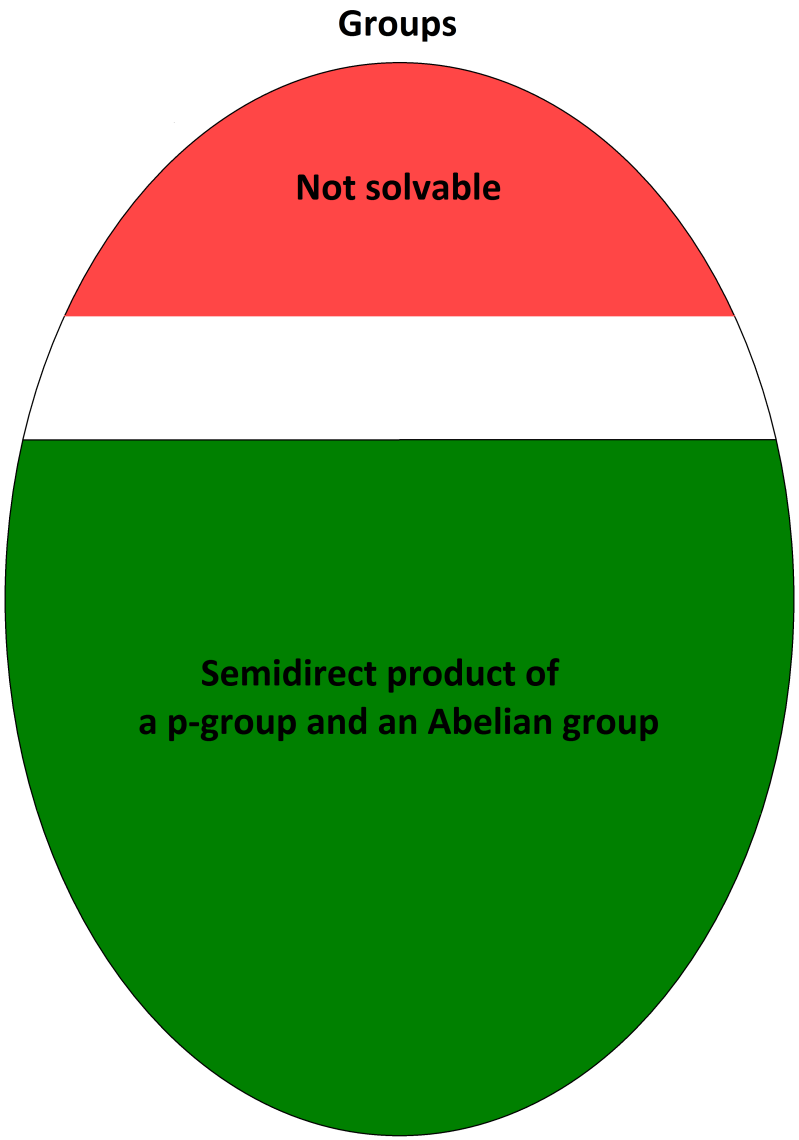


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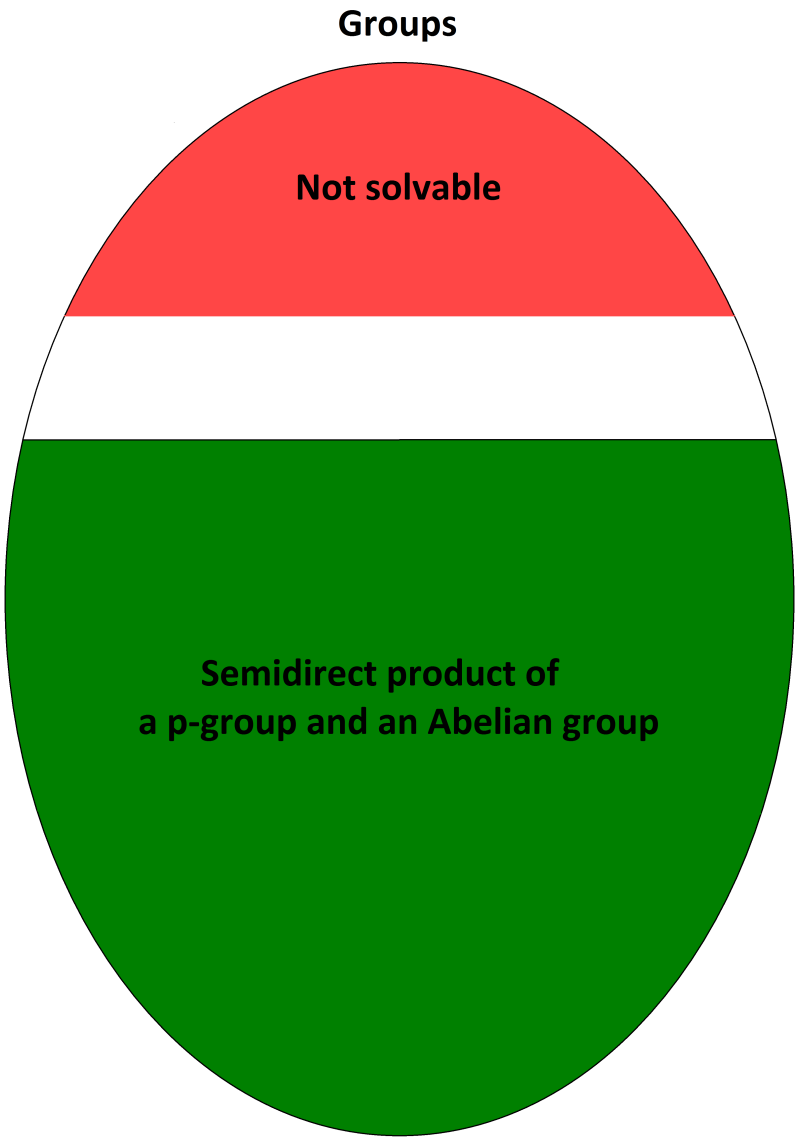


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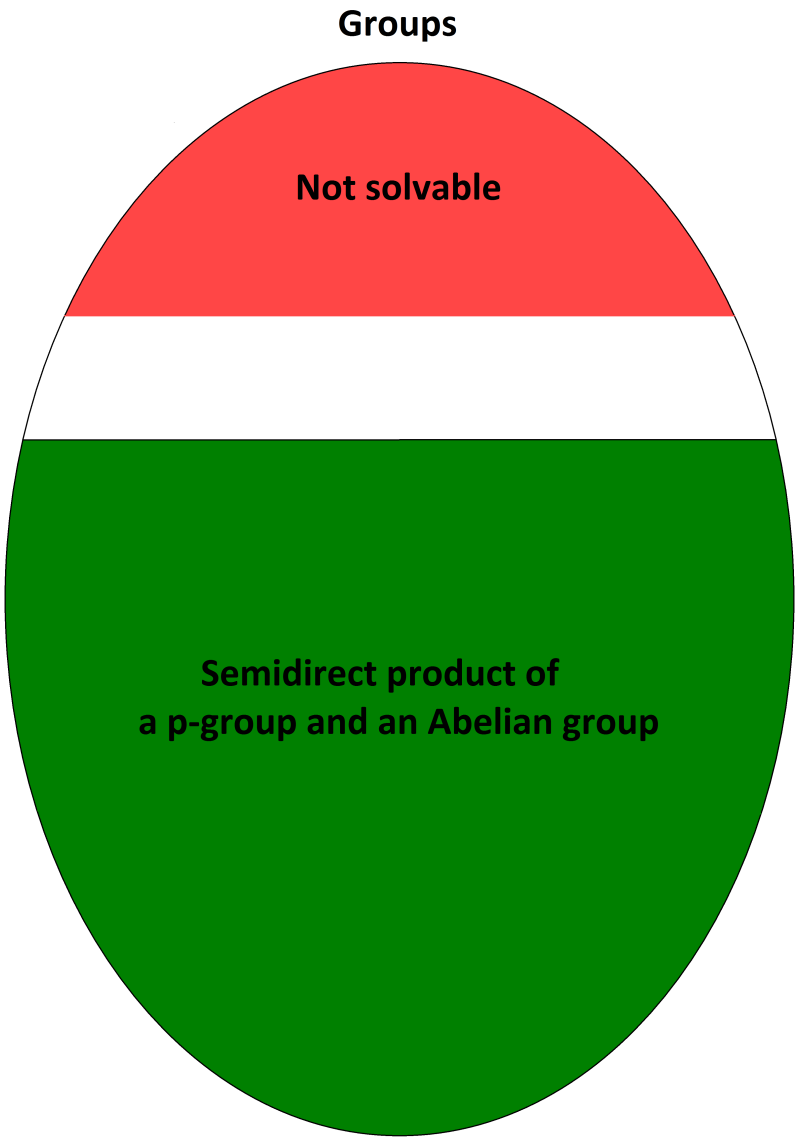


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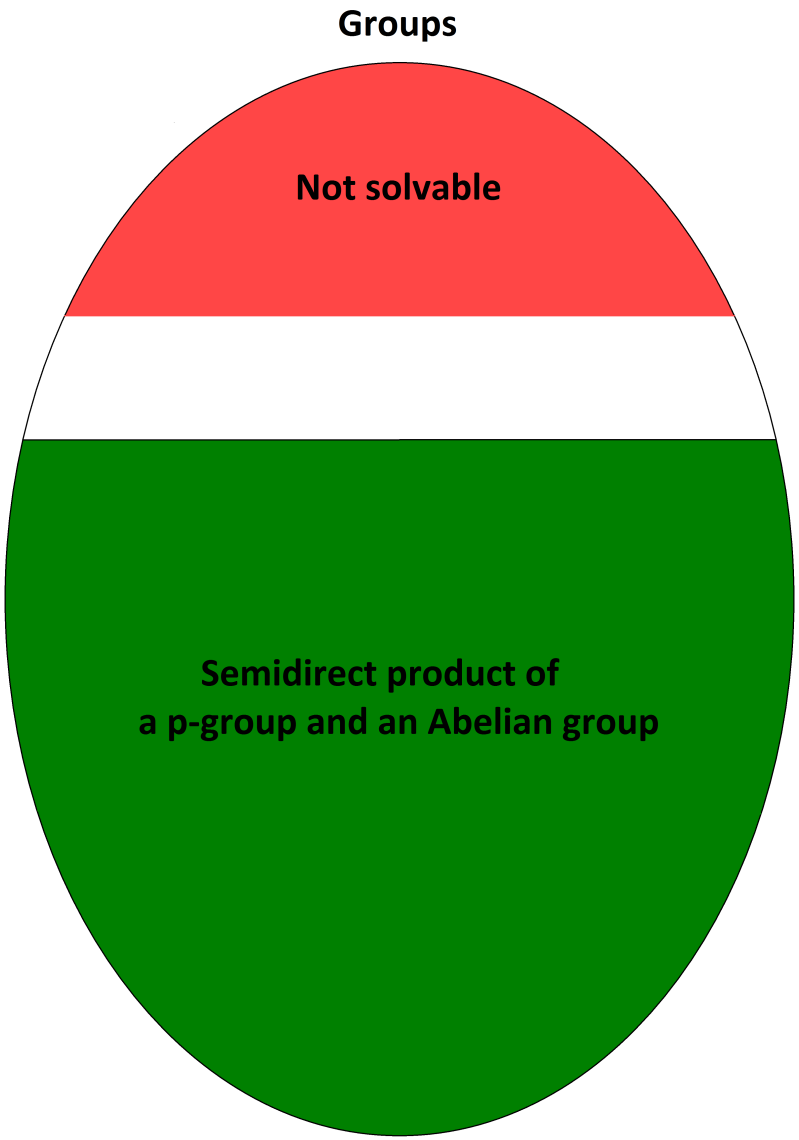


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## Thank you for your attention!

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$\implies$ the lengths of $f_1(\dots)$ and $f_2(\dots)$ is exponential in $n$