Spectra of commutative BCK-algebras

Matt Evans

Binghamton University
evans@math.binghamton.edu

BLAST 2019
University of Colorado, Boulder

May 23, 2019

Overview

cBCK-algebras

A **commutative BCK-algebra A** is an algebra $\mathbf{A} = \langle A; \cdot, 0 \rangle$ of type $(2, 0)$ such that

cBCK-algebras

A **commutative BCK-algebra A** is an algebra $\mathbf{A} = \langle A; \cdot, 0\rangle$ of type $(2, 0)$ such that

$$
\begin{aligned}
(x \cdot y) \cdot z &= (x \cdot z) \cdot y \\
x \cdot (x \cdot y) &= y \cdot (y \cdot x) \\
x \cdot x &= 0 \\
x \cdot 0 &= x
\end{aligned}
$$

for all $x, y, z \in A$.

cBCK-algebras

A **commutative BCK-algebra A** is an algebra $\mathbf{A} = \langle A; \cdot, 0\rangle$ of type $(2, 0)$ such that

$$
\begin{aligned}
(x \cdot y) \cdot z &= (x \cdot z) \cdot y \\
x \cdot (x \cdot y) &= y \cdot (y \cdot x) \\
x \cdot x &= 0 \\
x \cdot 0 &= x
\end{aligned}
$$

for all $x, y, z \in A$.

Facts

1 Every BCK-algebra is partially ordered via: $x \leq y$ iff $x \cdot y = 0$.

Facts

1 Every BCK-algebra is partially ordered via: $x \leq y$ iff $x \cdot y = 0$.

2 Commutativity: $x \wedge y := y \cdot (y \cdot x)$

Facts

1 Every BCK-algebra is partially ordered via: $x \leq y$ iff $x \cdot y = 0$.

2 Commutativity: $x \wedge y := y \cdot (y \cdot x) = x \cdot (x \cdot y) = y \wedge x$.

Facts

1 Every BCK-algebra is partially ordered via: $x \leq y$ iff $x \cdot y = 0$.

2 Commutativity: $x \wedge y := y \cdot (y \cdot x) = x \cdot (x \cdot y) = y \wedge x$.

A BCK-algebra **A** is **bounded** if there exists an element $1 \in A$ such that $x \cdot 1 = 0$ for all $x \in A$.

Facts

1 Every BCK-algebra is partially ordered via: $x \leq y$ iff $x \cdot y = 0$.

2 Commutativity: $x \wedge y := y \cdot (y \cdot x) = x \cdot (x \cdot y) = y \wedge x$.

A BCK-algebra $\mathbf{A}$ is **bounded** if there exists an element $1 \in A$ such that $x \cdot 1 = 0$ for all $x \in A$.

3 If $\mathbf{A}$ is bounded, then $\langle A; \wedge, \vee \rangle$ is a distributive lattice, where

$$x \vee y := 1 \cdot \left[(1 \cdot x) \wedge (1 \cdot y) \right].$$

(Traczyk, 1979)

Examples

1. Let X be a set. Then $\langle \mathcal{P}(X); -, \varnothing \rangle$ is a cBCK-algebra.

Examples

1. Let X be a set. Then $\langle \mathcal{P}(X); -, \varnothing \rangle$ is a cBCK-algebra.
2. More generally, any Boolean algebra $\mathbf{B}$ admits a cBCK-structure via $x \cdot y = x \wedge (\neg y)$.

Examples

1. Let X be a set. Then $\left\langle \mathcal{P}(X); -, \varnothing \right\rangle$ is a cBCK-algebra.
2. More generally, any Boolean algebra $\mathbf{B}$ admits a cBCK-structure via $x \cdot y = x \wedge (\neg y)$.
3. $\mathbf{R}_{\geq 0} = \langle \mathbb{R}_{\geq 0}; \cdot, 0 \rangle$ is a cBCK-algebra, where $x \cdot y = \max\{x - y, 0\}$.

Examples

1. Let X be a set. Then $\langle \mathcal{P}(X); -, \varnothing \rangle$ is a cBCK-algebra.
2. More generally, any Boolean algebra $\mathbf{B}$ admits a cBCK-structure via $x \cdot y = x \wedge (\neg y)$.
3. $\mathbf{R}_{\geq 0} = \langle \mathbb{R}_{\geq 0}; \cdot, 0 \rangle$ is a cBCK-algebra, where $x \cdot y = \max\{x - y, 0\}$.
4. $\mathbb{N}_0$ is a subalgebra of $\mathbf{R}_{\geq 0}$

Examples

1. Let X be a set. Then $\left\langle \mathcal{P}(X); -, \varnothing \right\rangle$ is a cBCK-algebra.
2. More generally, any Boolean algebra $\mathbf{B}$ admits a cBCK-structure via $x \cdot y = x \wedge (\neg y)$.
3. $\mathbf{R}_{\geq 0} = \langle \mathbb{R}_{\geq 0}; \cdot, 0 \rangle$ is a cBCK-algebra, where $x \cdot y = \max\{x - y, 0\}$.
4. $\mathbb{N}_0$ is a subalgebra of $\mathbf{R}_{\geq 0}$
5. Let $\mathcal{O} = \mathbb{N}_0 \overline{\oplus} \mathbb{N}_0^{\partial}$ denote the ordinal sum of $\mathbb{N}_0$ with its order-dual. Then $\mathcal{O}$ admits a cBCK-structure.

Example

Let $\{\mathbf{A}_\lambda\}_{\lambda \in \Lambda}$ be a family of cBCK-algebras with $A_\lambda \cap A_\mu = \{0\}$.

Example

Let $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of cBCK-algebras with $A_\lambda \cap A_\mu = \{0\}$.

Put $\mathcal{U} = \bigcup_{\lambda\in\Lambda} A_\lambda$.

Example

Let $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of cBCK-algebras with $A_\lambda \cap A_\mu = \{0\}$.

Put $\mathcal{U} = \bigcup_{\lambda\in\Lambda} A_\lambda$.

Then $\mathcal{U}$ is a cBCK-algebra with the operation

$$x \cdot y = \begin{cases} x \cdot_\lambda y & \text{if } x, y \in A_\lambda \\ x & \text{otherwise .} \end{cases}$$

Example

Let $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of cBCK-algebras with $A_\lambda \cap A_\mu = \{0\}$.

Put $\mathcal{U} = \bigcup_{\lambda\in\Lambda} A_\lambda$.

Then $\mathcal{U}$ is a cBCK-algebra with the operation

$$x \cdot y = \begin{cases} x \cdot_\lambda y & \text{if } x, y \in A_\lambda \\ x & \text{otherwise .} \end{cases}$$

We call $\mathcal{U}$ the **BCK-union** of the $\mathbf{A}_\lambda$'s and we will write $\mathcal{U} = \bigsqcup \mathbf{A}_\lambda$

Ideals

Given a cBCK-algebra **A**, a subset I is an **ideal** if

1. $0 \in I$
2. if $x \cdot y \in I$ and $y \in I$, then $x \in I$.

Ideals

Given a cBCK-algebra $\mathbf{A}$, a subset I is an **ideal** if

1. $0 \in I$
2. if $x \cdot y \in I$ and $y \in I$, then $x \in I$.

A proper ideal P of $\mathbf{A}$ is a **prime ideal** if $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

Ideals

Given a cBCK-algebra $\mathbf{A}$, a subset I is an **ideal** if

1. $0 \in I$
2. if $x \cdot y \in I$ and $y \in I$, then $x \in I$.

A proper ideal P of $\mathbf{A}$ is a **prime ideal** if $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

Let $\mathrm{Id}(\mathbf{A})$ and $\mathrm{X}(\mathbf{A})$ denote the set of ideals and prime ideals, respectively.

Examples

1. $\mathsf{Id}(\mathbb{N}_0) = \Big\{\{0\}, \mathbb{N}_0\Big\}$ and $\mathrm{X}(\mathbb{N}_0) = \{\{0\}\}$.

Examples

1. $\mathrm{Id}(\mathbb{N}_0) = \Big\{\{0\}, \mathbb{N}_0\Big\}$ and $\mathrm{X}(\mathbb{N}_0) = \{\{0\}\}$.
2. $\mathrm{Id}(\mathcal{O}) = \Big\{\{0\}, \mathbb{N}_0, \mathcal{O}\Big\}$ and $\mathrm{X}(\mathcal{O}) = \{\{0\}, \mathbb{N}_0, \}$.

Examples

1. $\mathsf{Id}(\mathbb{N}_0) = \Big\{\{0\}, \mathbb{N}_0\Big\}$ and $\mathrm{X}(\mathbb{N}_0) = \{\{0\}\}$.
2. $\mathsf{Id}(\mathcal{O}) = \Big\{\{0\}, \mathbb{N}_0, \mathcal{O}\Big\}$ and $\mathrm{X}(\mathcal{O}) = \{\{0\}, \mathbb{N}_0, \}$.

Lemma

If $\mathcal{U} = \bigsqcup \mathbf{A}_\lambda$ is a BCK-union, then any ideal has the form $I = \bigsqcup I_\lambda$, where $I_\lambda \in Id(\mathbf{A}_\lambda)$.

Examples

1. $\mathrm{Id}(\mathbb{N}_0) = \Big\{\{0\}, \mathbb{N}_0\Big\}$ and $\mathrm{X}(\mathbb{N}_0) = \big\{\{0\}\big\}$.
2. $\mathrm{Id}(\mathcal{O}) = \Big\{\{0\}, \mathbb{N}_0, \mathcal{O}\Big\}$ and $\mathrm{X}(\mathcal{O}) = \big\{\{0\}, \mathbb{N}_0, \big\}$.

Lemma

If $\mathcal{U} = \bigsqcup \mathbf{A}_\lambda$ is a BCK-union, then any ideal has the form $I = \bigsqcup I_\lambda$, where $I_\lambda \in Id(\mathbf{A}_\lambda)$.

Theorem (E., 2019)

An ideal P in a BCK-union $\mathcal{U}$ is prime iff there exists $\mu \in \Lambda$ and $Q \in X(\mathbf{A}_\mu)$ such that

$$P = \bigsqcup \mathbf{A}^Q_{\lambda,\mu},$$

where $\mathbf{A}^Q_{\lambda,\mu} = \begin{cases} \mathbf{A}_\lambda & \text{if } \lambda \neq \mu \\ Q & \text{if } \lambda = \mu . \end{cases}$

Spectra

For a subset $S \subseteq \mathbf{A}$, define

$$\sigma(S) = \left\{ P \in \mathrm{X}(\mathbf{A}) \mid S \not\subseteq P \right\} .$$

Spectra

For a subset $S \subseteq \mathbf{A}$, define

$$\sigma(S) = \left\{ P \in \mathrm{X}(\mathbf{A}) \mid S \not\subseteq P \right\} .$$

We will write $\sigma(a)$ for $\sigma(\{a\})$.

Spectra

For a subset $S \subseteq \mathbf{A}$, define

$$\sigma(S) = \left\{ P \in \mathrm{X}(\mathbf{A}) \mid S \not\subseteq P \right\} .$$

We will write $\sigma(a)$ for $\sigma(\{a\})$.

Proposition

The family $\mathcal{T}(\mathbf{A}) = \left\{ \sigma(I) \mid I \in \mathrm{Id}(\mathbf{A}) \right\}$ is a topology on $\mathrm{X}(\mathbf{A})$,

Spectra

For a subset $S \subseteq \mathbf{A}$, define

$$\sigma(S) = \left\{ P \in \mathrm{X}(\mathbf{A}) \mid S \not\subseteq P \right\} .$$

We will write $\sigma(a)$ for $\sigma(\{a\})$.

Proposition

The family $\mathcal{T}(\mathbf{A}) = \left\{ \sigma(I) \mid I \in \mathrm{Id}(\mathbf{A}) \right\}$ is a topology on $\mathrm{X}(\mathbf{A})$, and $\mathcal{T}_0(\mathbf{A}) = \left\{ \sigma(a) \mid a \in A \right\}$ is a basis for this topology.

Spectra

For a subset $S \subseteq \mathbf{A}$, define

$$\sigma(S) = \left\{ P \in \mathrm{X}(\mathbf{A}) \mid S \not\subseteq P \right\} .$$

We will write $\sigma(a)$ for $\sigma(\{a\})$.

Proposition

The family $\mathcal{T}(\mathbf{A}) = \left\{ \sigma(I) \mid I \in \mathrm{Id}(\mathbf{A}) \right\}$ is a topology on $\mathrm{X}(\mathbf{A})$, and $\mathcal{T}_0(\mathbf{A}) = \left\{ \sigma(a) \mid a \in A \right\}$ is a basis for this topology.

The space $\left(\mathrm{X}(\mathbf{A}) , \mathcal{T}(\mathbf{A}) \right)$ is the **spectrum** of $\mathbf{A}$.

A topological space $(X, \mathcal{T})$ is a **Stone space** if it is compact, Hausdorff, and totally disconnected.

A topological space $(X, \mathcal{T})$ is a **Stone space** if it is compact, Hausdorff, and totally disconnected.

An ordered topological space $(X, \mathcal{T}, \leq)$ is called a **Priestley space** if $(X, \mathcal{T})$ is a Stone space and $(X, \leq)$ satisfies the following separation property:

if $x \not\leq y$, there exists a clopen up-set U such that $x \in U$ but $y \notin U$.

A topological space $(X, \mathcal{T})$ is a **Stone space** if it is compact, Hausdorff, and totally disconnected.

An ordered topological space $(X, \mathcal{T}, \leq)$ is called a **Priestley space** if $(X, \mathcal{T})$ is a Stone space and $(X, \leq)$ satisfies the following separation property:

if $x \not\leq y$, there exists a clopen up-set U such that $x \in U$ but $y \notin U$.

Theorem (Meng & Jun, 1998)

If $\mathbf{A}$ is a bounded cBCK-algebra, then $\mathrm{X}(\mathbf{A})$ is a Stone space.

Theorem (E., 2017)

If **A** is a bounded and involutory cBCK-algebra, then $\mathrm{X}(\mathbf{A})$ is a Priestley space.

Theorem (E., 2017)

If $\mathbf{A}$ is a bounded and involutory cBCK-algebra, then $\mathrm{X}(\mathbf{A})$ is a Priestley space.

A BCK-algebra $\mathbf{A}$ is **involutory** if, for every pair of elements $x, y \in A$, the decreasing sequence

$$x \geq x \cdot y \geq (x \cdot y) \cdot y \geq \big((x \cdot y) \cdot y\big) \cdot y \geq \cdots$$

eventually stabilizes.

Theorem (E., 2017)

If **A** is a bounded and involutory cBCK-algebra, then $\mathrm{X}(\mathbf{A})$ is a Priestley space.

A BCK-algebra **A** is **involutory** if, for every pair of elements $x, y \in A$, the decreasing sequence

$$x \geq x \cdot y \geq (x \cdot y) \cdot y \geq \big((x \cdot y) \cdot y\big) \cdot y \geq \cdots$$

eventually stabilizes.

Proposition (E., 2018)

$\mathrm{X}(\mathbf{A})$ is locally compact for any cBCK-algebra **A**.

Theorem (E., 2017)

If **A** is a bounded and involutory cBCK-algebra, then $\mathrm{X}(\mathbf{A})$ is a Priestley space.

A BCK-algebra **A** is **involutory** if, for every pair of elements $x, y \in A$, the decreasing sequence

$$x \geq x \cdot y \geq (x \cdot y) \cdot y \geq \big((x \cdot y) \cdot y\big) \cdot y \geq \cdots$$

eventually stabilizes.

Proposition (E., 2018)

$\mathrm{X}(\mathbf{A})$ is locally compact for any cBCK-algebra **A**.

Comm	Comm, Bdd	Comm, Invol	Comm, Bdd, Invol
locally compact	compact Hausdorff t.d.	locally compact Hausdorff t.d. Pries. sep.	compact Hausdorff t.d. Pries. sep.

Comm	Comm, Bdd	Comm, Invol	Comm, Bdd, Invol
locally compact	compact Hausdorff t.d.	locally compact Hausdorff t.d. Pries. sep.	compact Hausdorff t.d. Pries. sep.

Comm	Comm, Bdd	Comm, Invol	Comm, Bdd, Invol
locally compact	compact Hausdorff t.d.	locally compact Hausdorff t.d. Pries. sep.	compact Hausdorff t.d. Pries. sep.

Open Question

Is **A** being involutory a necessary condition for $\mathrm{X}(\mathbf{A})$ to satisfy Priestley separation?

Comm	Comm, Bdd	Comm, Invol	Comm, Bdd, Invol
locally compact	compact Hausdorff t.d.	locally compact Hausdorff t.d. Pries. sep.	compact Hausdorff t.d. Pries. sep.

Open Question

Is **A** being involutory a necessary condition for $\mathrm{X}(\mathbf{A})$ to satisfy Priestley separation?

Open Question

What is the precise relationship between boundedness of **A** and compactness of $\mathrm{X}(\mathbf{A})$?

Comm	Comm, Bdd	Comm, Invol	Comm, Bdd, Invol
locally compact	compact Hausdorff t.d.	locally compact Hausdorff t.d. Pries. sep.	compact Hausdorff t.d. Pries. sep.

Open Question

Is **A** being involutory a necessary condition for $\mathrm{X}(\mathbf{A})$ to satisfy Priestley separation?

Open Question

What is the precise relationship between boundedness of **A** and compactness of $\mathrm{X}(\mathbf{A})$? Specifically, is there an unbounded **A** with infinite compact spectrum?

Comm	Comm, Bdd	Comm, Invol	Comm, Bdd, Invol
locally compact	compact Hausdorff t.d.	locally compact Hausdorff t.d. Pries. sep.	compact Hausdorff t.d. Pries. sep.

Open Question

Is **A** being involutory a necessary condition for $\mathrm{X}(\mathbf{A})$ to satisfy Priestley separation?

Open Question

What is the precise relationship between boundedness of **A** and compactness of $\mathrm{X}(\mathbf{A})$? Specifically, is there an unbounded **A** with infinite compact spectrum?

Open Question

What is the image of the functor $\mathrm{X} : \mathbf{cBCK} \to \mathbf{Top}$?

Examples

1. $\mathrm{X}(\mathbb{N}_0)$ is a one-point space.

Examples

1. $\mathrm{X}\left(\mathbb{N}_0\right)$ is a one-point space.
2. $\mathrm{X}\left(\mathcal{O}\right)$ is the Sierpinski space. [Reminder: $\mathcal{O} = \mathbb{N}_0 \overline{\oplus}\, \mathbb{N}_0^{\partial}$.]

Examples

1. $\mathrm{X}\big(\mathbb{N}_0\big)$ is a one-point space.
2. $\mathrm{X}\big(\mathcal{O}\big)$ is the Sierpinski space. [Reminder: $\mathcal{O} = \mathbb{N}_0 \overline{\oplus}\, \mathbb{N}_0^{\partial}$.]
3. $\mathrm{X}\Big(\bigsqcup_{\lambda\in\Lambda} \mathbb{N}_0\Big)$ is a discrete space of cardinality $|\Lambda|$.

Examples

1. $\mathrm{X}(\mathbb{N}_0)$ is a one-point space.

2. $\mathrm{X}(\mathcal{O})$ is the Sierpinski space. [Reminder: $\mathcal{O} = \mathbb{N}_0 \overline{\oplus} \mathbb{N}_0^\partial$.]

3. $\mathrm{X}\left(\bigsqcup_{\lambda\in\Lambda} \mathbb{N}_0\right)$ is a discrete space of cardinality $|\Lambda|$.

4. $\mathrm{X}\left(\bigsqcup_{\lambda\in\Lambda} \mathcal{O}\right)$ is actually interesting!

Example: $\mathcal{O}_1 \bigsqcup \mathcal{O}_2$

Consider $\mathbf{A} = \mathcal{O}_1 \bigsqcup \mathcal{O}_2$ and for brevity let $\mathrm{X} = \mathrm{X}(\mathbf{A})$. The prime ideals of $\mathbf{A}$ are:

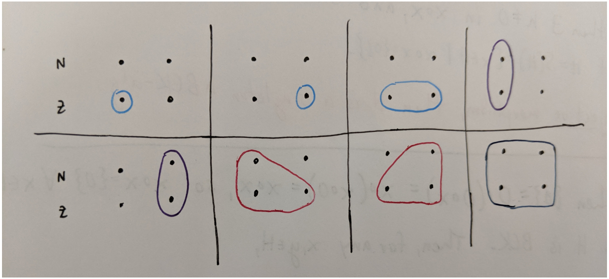
$$Z_1 = \{0\} \sqcup \mathcal{O}_2 \quad N_1 = \mathbb{N}_0 \sqcup \mathcal{O}_2$$

Example: $\mathcal{O}_1 \bigsqcup \mathcal{O}_2$

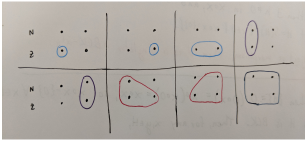
Consider $\mathbf{A} = \mathcal{O}_1 \bigsqcup \mathcal{O}_2$ and for brevity let $\mathrm{X} = \mathrm{X}(\mathbf{A})$. The prime ideals of $\mathbf{A}$ are:

$$
\begin{aligned}
Z_1 &= \{0\} \sqcup \mathcal{O}_2 \quad & N_1 &= \mathbb{N}_0 \sqcup \mathcal{O}_2 \\
Z_2 &= \mathcal{O}_1 \sqcup \{0\} \quad & N_2 &= \mathcal{O}_1 \sqcup \mathbb{N}_0
\end{aligned}
$$

So X is a 4-point space and the topology is:



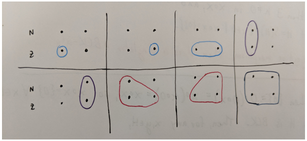
Example: $\mathcal{O}_1 \bigsqcup \mathcal{O}_2$



Some facts:

1. X is not Hausdorff.

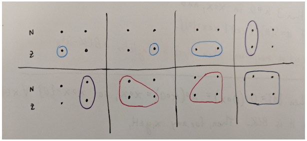
Example: $\mathcal{O}_1 \bigsqcup \mathcal{O}_2$



Some facts:

1. X is not Hausdorff.
2. Every irreducible closed set is the closure of a unique point; that is, X is a sober space.

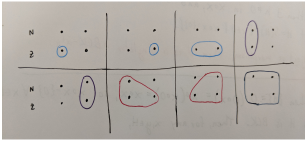
Example: $\mathcal{O}_1 \bigsqcup \mathcal{O}_2$



Some facts:

1. X is not Hausdorff.
2. Every irreducible closed set is the closure of a unique point; that is, X is a sober space.
3. The compact open sets form a basis and are closed under finite intersections.

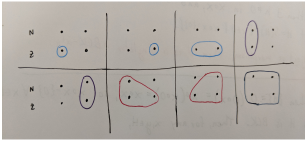
Example: $\mathcal{O}_1 \bigsqcup \mathcal{O}_2$



Some facts:

1. X is not Hausdorff.
2. Every irreducible closed set is the closure of a unique point; that is, X is a sober space.
3. The compact open sets form a basis and are closed under finite intersections.
4. X is compact.

Example: $\mathcal{O}_1 \bigsqcup \mathcal{O}_2$

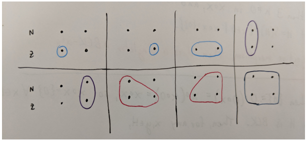


Some facts:

1. X is not Hausdorff.
2. Every irreducible closed set is the closure of a unique point; that is, X is a sober space.
3. The compact open sets form a basis and are closed under finite intersections.
4. X is compact.

Properties (2)-(4) tell us that X is a **spectral space**

Example: $\mathcal{O}_1 \bigsqcup \mathcal{O}_2$



Some facts:

1. X is not Hausdorff.
2. Every irreducible closed set is the closure of a unique point; that is, X is a sober space.
3. The compact open sets form a basis and are closed under finite intersections.
4. X is compact.

Properties (2)-(4) tell us that X is a **spectral space** ; that is, X is homeomorphic to the spectrum of some commutative ring.

Examples: $\bigsqcup_{i=1}^{n} \mathcal{O}_i$ and $\bigsqcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$

More generally, the spectrum of $\bigsqcup_{i=1}^{n} \mathcal{O}_i$ is a non-Hausdorff spectral space with cardinality $2n$.

Examples: $\coprod_{i=1}^{n} \mathcal{O}_i$ and $\coprod_{\lambda\in\Lambda} \mathcal{O}_\lambda$

More generally, the spectrum of $\coprod_{i=1}^{n} \mathcal{O}_i$ is a non-Hausdorff spectral space with cardinality $2n$.

If we let Λ be an infinite indexing set, then the spectrum of $\coprod_{\lambda\in\Lambda} \mathcal{O}_\lambda$ is no longer compact, but still has properties (2) and (3).

Examples: $\bigsqcup_{i=1}^{n} \mathcal{O}_i$ and $\bigsqcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$

More generally, the spectrum of $\bigsqcup_{i=1}^{n} \mathcal{O}_i$ is a non-Hausdorff spectral space with cardinality $2n$.

If we let Λ be an infinite indexing set, then the spectrum of $\bigsqcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$ is no longer compact, but still has properties (2) and (3). This makes it a **generalized spectral space**, meaning it is homeomorphic to the spectrum of some distributive lattice with zero.

Examples: $\bigsqcup_{i=1}^{n} \mathcal{O}_i$ and $\bigsqcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$

More generally, the spectrum of $\bigsqcup_{i=1}^{n} \mathcal{O}_i$ is a non-Hausdorff spectral space with cardinality $2n$.

If we let Λ be an infinite indexing set, then the spectrum of $\bigsqcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$ is no longer compact, but still has properties (2) and (3). This makes it a **generalized spectral space**, meaning it is homeomorphic to the spectrum of some distributive lattice with zero.

Some associated facts:

Some associated facts:

Proposition (E., 2019)

$\mathrm{X}\left(\bigsqcup \mathbf{A}_\lambda\right) \cong \bigsqcup \mathrm{X}(\mathbf{A}_\lambda)$ with the disjoint union topology.

Some associated facts:

Proposition (E., 2019)

$\mathrm{X}\left(\bigsqcup \mathbf{A}_\lambda\right) \cong \bigsqcup \mathrm{X}(\mathbf{A}_\lambda)$ with the disjoint union topology.

Proposition (E., 2019)

$$\mathrm{X}\left(\prod_{i=1}^{n} \mathbf{A}_i\right) \cong \mathrm{X}\left(\bigsqcup_{i=1}^{n} \mathbf{A}_i\right)$$

Connected Unions

A family $\{\mathbf{A}_\lambda\}_{\lambda \in \Lambda}$ of BCK-algebras is **connected** if for all $\lambda, \mu \in \Lambda$:

1. The operations $\cdot_\lambda$ and $\cdot_\mu$ coincide on $\mathbf{A}_\lambda \cap \mathbf{A}_\mu$, and

Connected Unions

A family $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ of BCK-algebras is **connected** if for all $\lambda, \mu \in \Lambda$:

1. The operations $\cdot_\lambda$ and $\cdot_\mu$ coincide on $\mathbf{A}_\lambda \cap \mathbf{A}_\mu$, and
2. if $x \in \mathbf{A}_\lambda \setminus \mathbf{A}_\mu$ and $y \in \mathbf{A}_\mu \setminus \mathbf{A}_\lambda$, then $\text{glb}\{x, y\} \in \mathbf{A}_\lambda \cap \mathbf{A}_\mu$.

Connected Unions

A family $\{\mathbf{A}_\lambda\}_{\lambda \in \Lambda}$ of BCK-algebras is **connected** if for all $\lambda, \mu \in \Lambda$:

1. The operations $\cdot_\lambda$ and $\cdot_\mu$ coincide on $\mathbf{A}_\lambda \cap \mathbf{A}_\mu$, and
2. if $x \in \mathbf{A}_\lambda \setminus \mathbf{A}_\mu$ and $y \in \mathbf{A}_\mu \setminus \mathbf{A}_\lambda$, then $\text{glb}\{x, y\} \in \mathbf{A}_\lambda \cap \mathbf{A}_\mu$.

Given a connected family $\mathcal{A} = \{\mathbf{A}_\lambda\}_{\lambda \in \Lambda}$, we say $\mathbf{A} = \langle A; \cdot, 0 \rangle$ is a **connected BCK-union** of the family $\mathcal{A}$ provided

1. $A = \bigcup_{\lambda \in \Lambda} A_\lambda$, and

Connected Unions

A family $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ of BCK-algebras is **connected** if for all $\lambda, \mu \in \Lambda$:

1. The operations $\cdot_\lambda$ and $\cdot_\mu$ coincide on $\mathbf{A}_\lambda \cap \mathbf{A}_\mu$, and
2. if $x \in \mathbf{A}_\lambda \setminus \mathbf{A}_\mu$ and $y \in \mathbf{A}_\mu \setminus \mathbf{A}_\lambda$, then $\mathrm{glb}\{x, y\} \in \mathbf{A}_\lambda \cap \mathbf{A}_\mu$.

Given a connected family $\mathcal{A} = \{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$, we say $\mathbf{A} = \langle A; \cdot, 0\rangle$ is a **connected BCK-union** of the family $\mathcal{A}$ provided

1. $A = \bigcup_{\lambda\in\Lambda} A_\lambda$, and
2. the operation $\cdot$ agrees with $\cdot_\lambda$ when restricted to $\mathbf{A}_\lambda$.

Connected Unions

A family $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ of BCK-algebras is **connected** if for all $\lambda, \mu \in \Lambda$:

1. The operations $\cdot_\lambda$ and $\cdot_\mu$ coincide on $\mathbf{A}_\lambda \cap \mathbf{A}_\mu$, and
2. if $x \in \mathbf{A}_\lambda \setminus \mathbf{A}_\mu$ and $y \in \mathbf{A}_\mu \setminus \mathbf{A}_\lambda$, then $\mathrm{glb}\{x, y\} \in \mathbf{A}_\lambda \cap \mathbf{A}_\mu$.

Given a connected family $\mathcal{A} = \{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$, we say $\mathbf{A} = \langle A; \cdot, 0\rangle$ is a **connected BCK-union** of the family $\mathcal{A}$ provided

1. $A = \bigcup_{\lambda\in\Lambda} A_\lambda$, and
2. the operation $\cdot$ agrees with $\cdot_\lambda$ when restricted to $\mathbf{A}_\lambda$.

Note: the BCK-union we defined earlier is a special case of connected BCK-union with $\mathbf{A}_\lambda \cap \mathbf{A}_\mu = \{0\}$ for all pairs of indices.

Connected Unions

A family $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ of BCK-algebras is **connected** if for all $\lambda, \mu \in \Lambda$:

1. The operations $\cdot_\lambda$ and $\cdot_\mu$ coincide on $\mathbf{A}_\lambda \cap \mathbf{A}_\mu$, and
2. if $x \in \mathbf{A}_\lambda \setminus \mathbf{A}_\mu$ and $y \in \mathbf{A}_\mu \setminus \mathbf{A}_\lambda$, then $\mathrm{glb}\{x, y\} \in \mathbf{A}_\lambda \cap \mathbf{A}_\mu$.

Given a connected family $\mathcal{A} = \{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$, we say $\mathbf{A} = \langle A; \cdot, 0\rangle$ is a **connected BCK-union** of the family $\mathcal{A}$ provided

1. $A = \bigcup_{\lambda\in\Lambda} A_\lambda$, and
2. the operation $\cdot$ agrees with $\cdot_\lambda$ when restricted to $\mathbf{A}_\lambda$.

Note: the BCK-union we defined earlier is a special case of connected BCK-union with $\mathbf{A}_\lambda \cap \mathbf{A}_\mu = \{0\}$ for all pairs of indices.

Note: Any family of *commutative* BCK-algebras satisfying (1) automatically satisfies (2).

Connected Unions

Theorem (Pałasinski, 1982)

1. *The connected BCK-union for every connected family of BCK-algebras exists and it is unique.*

Connected Unions

Theorem (Pałasinski, 1982)

1. *The connected BCK-union for every connected family of BCK-algebras exists and it is unique.*
2. *If* $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ *is a connected family of commutative BCK-algebras, then their connected BCK-union is a commutative BCK-algebra.*

Connected Unions

Theorem (Pałasinski, 1982)

1. *The connected BCK-union for every connected family of BCK-algebras exists and it is unique.*
2. *If* $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ *is a connected family of commutative BCK-algebras, then their connected BCK-union is a commutative BCK-algebra.*
3. *Every cBCK-algebra is the connected BCK-union of a connected family of directed cBCK-algebras.*

Connected Unions

Theorem (Pałasinski, 1982)

1. *The connected BCK-union for every connected family of BCK-algebras exists and it is unique.*
2. *If $\{\mathbf{A}_\lambda\}_{\lambda\in\Lambda}$ is a connected family of commutative BCK-algebras, then their connected BCK-union is a commutative BCK-algebra.*
3. *Every cBCK-algebra is the connected BCK-union of a connected family of directed cBCK-algebras.*

Current/Future Work

Reminder:

Lemma

If $\mathcal{U} = \bigsqcup \mathbf{A}_\lambda$ is a BCK-union, then any ideal has the form $I = \bigsqcup I_\lambda$, where $I_\lambda \in Id(\mathbf{A}_\lambda)$.

Current/Future Work

Reminder:

Lemma

If $\mathcal{U} = \bigsqcup \mathbf{A}_\lambda$ *is a BCK-union, then any ideal has the form* $I = \bigsqcup I_\lambda$, *where* $I_\lambda \in Id(\mathbf{A}_\lambda)$.

Current Can we decompose the ideals of a connected BCK-union in some "nice" way that is similar to the BCK-union?

Current/Future Work

Reminder:

Lemma

If $\mathcal{U} = \bigsqcup \mathbf{A}_\lambda$ is a BCK-union, then any ideal has the form $I = \bigsqcup I_\lambda$, where $I_\lambda \in Id(\mathbf{A}_\lambda)$.

Current: Can we decompose the ideals of a connected BCK-union in some "nice" way that is similar to the BCK-union?

Future: Same question for prime ideals.

Thank you!

References

Iseki, Tanaka
An introduction to the theory of BCK-algebras
Mathematica Japonica, 1978, **23** (1), 1-26.

Meng, Jun
The spectral space of MV-algebras is a Stone space
Scientiae Mathematicae, 1998, **1** (2), 211-215.

Pałasinski
Representation theorem for commutative BCK-algebras
Mathematics Seminar Notes, Kobe University, 1982, **10**, 473-478.

Traczyk
On the variety of bounded commutative BCK-algebras
Mathematica Japonica, 1979, **24** (3), 283-292.