On extensions of partial isometries

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May 22, 2019

Overview

1. Herwig–Lascar extension theorem
2. S-extensions
3. Coherent S-extension and ultraextensive metric spaces
4. Generalization to finite relational languages
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6. Compact spaces

Herwig–Lascar extension theorem

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The main problem: extending the semigroup of partial automorphisms (partial symmetries) of a “nice” structure X to a group of automorphisms (full symmetries) of a“nice” extension of X, Y. We think of Y as a solution for X.

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One of the first results of this sort was proven by Hrushovski for finite (simple) graphs.

Theorem (Hrushovski)

Every finite (simple) graph G can be extended to another finite graph H such that every partial isomorphism of G extends to an (full) isomorphism of H.

Herwig–Lascar extension theorem

Definition

Let C_1, C_2 be two structures in a given finite relational language $\mathcal{L}$. A **partial isomorphism** from C_1 into C_2 is an isomorphism of a substructure of C_1 onto a substructure of C_2.

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Definition

Let $\mathcal{C}$ be a class of $\mathcal{L}-$structures (containing both finite and infinite structures). $\mathcal{C}$ is said to have **the extension property for partial automorphisms (EPPA)** if whenever C_1 and C_2 are structures in $\mathcal{C}$, C_1 is finite, $C_1 \subseteq C_2$, and every partial automorphism of C_1 extends to an automorphism of C_2, then there exist a finite structure C_3 in $\mathcal{C}$ such that every partial automorphism of C_1 extends to an automorphism of C_3.

Herwig–Lascar extension theorem

Definition

If M is an $\mathcal{L}$-structure and $\mathcal{T}$ a set of $\mathcal{L}$-structures, we say that M is $\mathcal{T}$-**free** if there is no structure $T \in \mathcal{T}$ and weak homomorphism $h : T \underset{w}{\rightarrow} M$.

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Theorem (Herwig–Lascar)

Let $\mathcal{L}$ be a finite relational language and $\mathcal{T}$ a finite set of finite $\mathcal{L}$-structures. Then, the class of $\mathcal{T}$-free $\mathcal{L}$-structures has the EPPA.

Herwig–Lascar extension theorem

Theorem (Solecki)

Let X be a finite metric space. There exists a finite metric space Y such that X isometrically embeds into Y and every partial isometry of X extends to a full isometry of Y.

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This was also (re)proved by Pestov and Vershik, independently. Also, Jan Hubička, Matěj Konečnỳ, and Jaroslav Nešetřil recently presented a combinatorial proof of Solecki's and HL's theorems.

S-extensions

S-extensions

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Definition

Let X be a metric space. An *S-extension* of X is a pair (Y, ϕ), where $Y \supseteq X$ is an extension of X, and $\phi : \mathscr{P}_X \to \mathrm{Iso}(Y)$ such that $\phi(p)$ extends p for all $p \in \mathscr{P}_X$. The map ϕ is called an *S-map* for X.

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Definition

Fix $a_0 \in X$. Let H be the set of all finite words $p_1 \cdots p_n$ with $p_1, \ldots, p_n \in \mathscr{P}_X$ such that $p_1 \ldots p_n(a_0) = p_1(p_2(\cdots p_n(a_0)\cdots))$ is defined and $p_1 \ldots p_n(a_0) = a_0$. Since X is finite, H is a finitely generated subgroup of $\mathbb{F}(\mathscr{P}_X)$.

S-extensions

Next we try to characterize all finite S-extensions.

Definition

An S-extension (Y, ϕ) of X is said to be *minimal* if for any $y \in Y$ there is $g \in \mathbb{F}(\mathscr{P}_X)$ such that $y = \phi(g)(a_0)$.

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Definition

Define $\Gamma = \mathbb{F}(\mathscr{P}_X)/H$. We construct a weighted graph (Γ, w) as follows:

1. for every $p, q \in \mathscr{P}_X \cup \{1\}$ such that $p(a_0)$ and $q(a_0)$ are defined, there is an edge between pH and qH with $w(pH, qH) = d_X(p(a_0), q(a_0))$, and
2. for every $g, g_1, g_2 \in \mathbb{F}(\mathscr{P}_X)$, if there is an edge between $g_1 H$ and $g_2 H$, then there is an edge between $gg_1 H$ and $gg_2 H$ with $w(gg_1 H, gg_2 H) = w(g_1 H, g_2 H)$.

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Theorem (E., Gao)

Let (Y,ϕ) be a finite minimal S-extension of X. Let $N = \ker(\phi)$ and $G = \Phi_N(\mathbb{F}(\mathscr{P}_X))$. Then there is a G-invariant pseudometric ρ on Γ_N which is consistent with w_N such that (Y,ϕ) is isomorphic to $(\overline{\Gamma_N}, \overline{\Phi_N})$.

Coherent S-extension and ultraextensive metric spaces

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Definition

Let $X_1 \subseteq X_2$ be metric spaces and (Y_i, ϕ_i) be an S-extension of X_i for $i = 1, 2$. We say that (Y_1, ϕ_1) and (Y_2, ϕ_2) are *coherent* if

(i) Y_2 extends Y_1,

(ii) $\phi_2(p)$ extends $\phi_1(p)$ for all $p \in \mathscr{P}_{X_1} \subseteq \mathscr{P}_{X_2}$, and

(iii) letting $K_i = \phi_i(\mathbb{F}(\mathscr{P}_{X_i})) \leq \mathrm{Iso}(Y_i)$ for $i = 1, 2$, and letting $\kappa : K_1 \to K_2$ be the unique group homomorphism such that $\kappa(\phi_1(p)) = \phi_2(p)$ for all $p \in \mathscr{P}_{X_1}$, then κ is a group isomorphic embedding from K_1 into K_2.

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Theorem (E., Gao)

Suppose $X_1 \subseteq X_2$ are finite metric spaces and (Y_1, ϕ_1) is a finite minimal S-extension of X_1. Then there is a finite minimal S-extension (Y_2, ϕ_2) of X_2 so that (Y_2, ϕ_2) is coherent with (Y_1, ϕ_1).

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Definition

A metric space U is *ultraextensive* if

(i) U is ultrahomogeneous, i.e., there is a ϕ such that (U, ϕ) is an S-extension of U;

(ii) Every finite $X \subseteq U$ has a finite S-extension (Y, ϕ) where $Y \subseteq U$;

(iii) If $X_1 \subseteq X_2 \subseteq U$ are finite and (Y_1, ϕ_1) is a finite minimal S-extension of X_1 with $Y_1 \subseteq U$, then there is a finite minimal S-extension (Y_2, ϕ_2) of X_2 such that $Y_2 \subseteq U$ and (Y_1, ϕ_1) and (Y_2, ϕ_2) are coherent.

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Theorem (E., Gao)

Every countable metric space can be extended to a countable ultraextensive metric space.

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If U is an ultraextensive metric space, then every countable subset $X \subseteq U$ can be extended to a countable ultraextensive $Y \subseteq U$. In particular, U has a dense ultraextensive subset.

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Solecki and Rosendal have proved this theorem for $\mathrm{Iso}(\mathbb{QU})$.

Generalization to finite relational languages

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Let $\mathcal{L}$ be a finite relational language.

Definition

Let C be an $\mathcal{L}$-structure. An *HL-extension* of C is a pair (D, ϕ), where $D \supseteq C$ is an extension of C, and $\phi : \mathscr{P}_C \to \mathrm{Iso}(D)$ such that $\phi(p)$ extends p for all $p \in \mathscr{P}_C$. The map ϕ is called an *HL-map* for C.

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An HL-extension (C_2, ϕ) of C_1 is said to be *minimal* if for any $b \in C_2$ there is $g \in \mathbb{F}(\mathscr{P}_{C_1})$ and $a \in C_1$such that $b = \phi(g)(a)$.

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Theorem (E., Gao)

Let $\mathcal{T}$ be a finite set of finite $\mathcal{L}$-structures and $\mathcal{C}$ the class of finite $\mathcal{T}$-free $\mathcal{L}$-structures. If $\mathcal{C}$ is a Fraisse class, then the Fraisse limit is ultraextensive.

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Let $\mathcal{T}$ be a finite set of finite $\mathcal{L}$-structures and $\mathcal{C}$ the class of finite $\mathcal{T}$-free $\mathcal{L}$-structures. If $\mathcal{C}$ is a Fraisse class, then the Fraisse limit is ultraextensive.

In particular, the universal triangle-free graph (Henson graph in general) is ultraextensive.

Compact ultrametric spaces

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Question [Solecki]

Let X be a compact metric space. Can we find a compact metric space Y such that X embeds isometrically into Y and every partial isometry of X extends to a full isometry of Y?

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We can answer the question for ultrametric compact spaces.

Theorem (E., Gao)

Let X be a finite ultrametric space. Then X can be extended to a finite ultraextensive ultrametric space Y. Furthermore, there is such Y so that the set of distances in X and Y are the same.

Compact ultrametric spaces

Theorem (E., Gao)

Let Y be a finite ultrametric space. Then the following are equivalent:

(i) *Y is homogeneous;*

(ii) *Y is ultrahomogeneous;*

(iii) *Y is ultraextensive.*

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Theorem (E., Gao)

Every compact ultrametric space can be extended to a compact ultraextensive ultrametric space. In particular, every compact ultrametric space has a compact ultrametric S-extension.

Compact metric spaces

First we consider extensions with ε-distortion.

Theorem (E., Gao)

Let X be a compact metric space. For every $\varepsilon > 0$ there exists a compact metric space, Y, such that X embeds isometrically into Y and for every partial isometry of X, p, there exists a full isometry of Y, q, such that

$$\forall x \in dom(p)\ (d_Y(p(x), q(x)) < \varepsilon)$$

Compact metric spaces

Theorem (E., Gao)

Let X be a compact metric space. Then the following are equivalent:

(a) *X has a compact S-extension;*

(b) *There is a triple (Y, G, φ) where Y is a compact metric space extending X, G is a compact subgroup of $\mathrm{Homeo}(Y)$, and φ is a map from the set of all partial isometries of X into G such that*

$$d(x,y) = d(\varphi(p)(x), \varphi(p)(y))$$

for all p, partial isometries of X, $x \in dom(X)$, and $y \in Y$.

Thank you!