A generalization of affine algebras

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- every f in F is affine with respect to $\mathbb{A}$.

Examples of affine algebras

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$\circ$	0	1	2
0	0	2	1
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Fundamental theorem of abelian algebras

Theorem (Herrmann, 1979)

Let $\mathbf{A}$ *belong to a modular variety. Then* $\mathbf{A}$ *is affine if and only if it is abelian.*

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Every algebra $\langle A; F \rangle$ gives rise to the clone $\mathrm{Clo}(A; F)$ generated by F. Two algebras $\langle A; F \rangle$ and $\langle A; G \rangle$ are term equivalent if and only if $\mathrm{Clo}(A; F) = \mathrm{Clo}(A; G)$.

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$$\mathcal{K}(\mathbb{A}, \mathbf{R}, \mathbf{M}) = \left\{ \sum_{i=1}^{n} r_i(x_i) + a : r_i \in R, (1 - \sum_{i=1}^{n} r_i, a) \in M \right\}$$

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Theorem (Szendrei, 1980)

If $\mathcal{C}$ is an affine clone on a set A, then there exist $\mathbb{A}$, $\mathbf{R}$, and $\mathbf{M}$ such that

$$\mathcal{C} = \mathcal{K}(\mathbb{A}, \mathbf{R}, \mathbf{M}).$$

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The *commutator of* α *and* β is

$$[\alpha, \beta] = \bigcap \{\gamma : \alpha \text{ centralizes } \beta \text{ modulo } \gamma\}.$$

Nilpotent algebras

An algebra **A** is *nilpotent* if there is a positive integer c and a sequence

$$1_A = \theta_0 \geq \theta_1 \geq \cdots \geq \theta_c = 0_A$$

of congruences satisfying $[1_A, \theta_i] \leq \theta_{i+1}$.

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- every f in F is nil-2-affine with respect to $\mathbb{A}$, θ.

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Relation to Mal'cev algebras of nilpotence class at most 2

Proposition (EC)

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Open Question

Does there exist an algebra **A** that is both Mal'cev and nilpotent of class 2, but is not nil-2-affine?

My main result (so far)

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Let $\mathbf{A} = \langle A; F \rangle$ *be an algebra such that* $x_1 - x_2 + x_3 \in \mathrm{Clo}(\mathbf{A})$ *for some abelian group* $\mathbb{A}$*. If*

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Is this true when θ is not a factor congruence?

A useful proposition

Let **A** be an algebra with Mal'cev term operation q and let α, β, γ be congruences of **A**. Define

$$C(q, \alpha, \beta, \gamma) = \{(x, y, z, w) \in A^4 : x \; \alpha \; y, y \; \beta \; z, q(x, y, z) \; \gamma \; w\}.$$

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Proposition (Aichinger & Mayr, 2007)

$[\alpha, \beta] \leq \gamma$ *if and only if* $C(q, \alpha, \beta, \gamma)$ *is a subuniverse of* $\mathbf{A}^4$*.*

Sketch of proof of my result

Suppose $[1_A, 1_A] \leq \theta$ and $[1_A, \theta] \leq 0_A$. So $C(x_1 - x_2 + x_3, 1_A, 1_A, \theta)$ and $C(x_1 - x_2 + x_3, 1_A, \theta, 0_A)$ are subuniverses of $\mathbf{A}^4$.

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$$g(x) = f^\circ(u) \text{ where } u \text{ is the unique elt of } U^n \text{ with } u \; \phi^n \; x$$

Sketch of proof of my result

Suppose $[1_A, 1_A] \leq \theta$ and $[1_A, \theta] \leq 0_A$. So $C(x_1 - x_2 + x_3, 1_A, 1_A, \theta)$ and $C(x_1 - x_2 + x_3, 1_A, \theta, 0_A)$ are subuniverses of $\mathbf{A}^4$. Let $f : A^n \to A$ be in F. Since θ is a factor congruence of $\mathbb{A}$, there is congruence ϕ of $\mathbb{A}$ such that $\theta \circ \phi = 1_A$ and $\theta \wedge \phi = 0_A$. Let $U = 0/\theta$ and $V = 0/\phi$. Define $f^\circ, g, h : A^n \to A$ by

$$f^\circ(x) = f(x) - f(0)$$

$$g(x) = f^\circ(u) \text{ where } u \text{ is the unique elt of } U^n \text{ with } u \ \phi^n \ x$$

$$h(x) \text{ is the unique elt of } V \text{ with } h(x) \ \theta \ f^\circ(x)$$

Sketch continued

For $1 \leq i \leq n$, define $r_i : A \to A$ by

$$r_i(x) = g(0, \dots, 0, \overset{i}{x}, 0, \dots, 0) + h(0, \dots, 0, \overset{i}{x}, 0, \dots, 0)$$

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Define $a : A^n \to A$ by

$$a(x) = f(x) - g(x) - h(x)$$

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Claim:

$$f(x_1, \dots, x_n) = r_1(x_1) + \cdots + r_n(x_n) + a(x_1, \dots, x_n)$$

where $r_1, \dots, r_n$ and a satisfy the desired properties.

Some useful propositions

Proposition (Exercise from Freese & McKenzie, 1987)

Every nilpotent Mal'cev algebra is polynomially equivalent to an expansion of a nilpotent loop.

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Every nilpotent Mal'cev algebra is polynomially equivalent to an expansion of a nilpotent loop.

Proposition (Daly & Vojtechovsky, 2009)

If a finite nilpotent loop has a central congruence of index 2, then it is an abelian group.

Nilpotent Mal'cev algebras of order $2p$

A (possibly true) proposition

Let **A** be a nilpotent Mal'cev algebra of order $2p$ for some odd prime p. Further suppose **A** has a central congruence θ of index 2. Then **A** is nil-2-affine.

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Proof.

Since **A** is nilpotent and Mal'cev, it is p.e. to an expansion **E** of a nilpotent loop $\mathbb{L}$.

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Proof.

Since **A** is nilpotent and Mal'cev, it is p.e. to an expansion **E** of a nilpotent loop $\mathbb{L}$. Since p.e. algebras share the same congruences and centrality relations, θ is a central congruence of $\mathbb{L}$.

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Since **A** is nilpotent and Mal'cev, it is p.e. to an expansion **E** of a nilpotent loop $\mathbb{L}$. Since p.e. algebras share the same congruences and centrality relations, θ is a central congruence of $\mathbb{L}$. So $\mathbb{L}$ is a cyclic group and θ is a factor congruence. If $x_1 - x_2 + x_3$ is a polynomial operation of **A**, then must it be a term operation? □

Nilpotent Mal'cev algebras of order pq

Open Question

Is every nilpotent loop of order pq (where p and q are distinct primes) nil-2-affine?

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Yes! At least when $pq = 6$.

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Another (possibly true) proposition

Let **A** be a nilpotent Mal'cev algebra of order pq where p and q are distinct primes. Further suppose a positive answer to the above question. Then **A** is nil-2-affine.

Future work

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- generalize Szendrei's result to nil-2-affine algebras

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- generalize Szendrei's result to nil-2-affine algebras
- extend to nil-c-affine algebras for $c > 2$.