Descriptive classification of abelian orbit equivalence relations

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A Polish space is a topological space which is separable and completely metrizable.

Polish spaces are the conventional spaces of analysis. But when dealing with questions of complexity, we usually do not care about any particular topology and instead consider only the Borel structure.

A standard Borel space is a set equipped with a σ-algebra of subsets which are the Borel sets given by a Polish topology.

- Any two uncountable Polish spaces are Borel isomorphic.
- So a set which is Borel in *some* Polish topology must be Borel in *all* Polish topologies.
- It makes sense to talk of *the* standard Borel structure of a space.

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G: a Polish or standard Borel group

X: a Polish or standard Borel space

$x \mapsto g \cdot x$: a continuous or Borel action

The orbit equivalence relation E_G^X:

$$xE_G^X y \Leftrightarrow \exists g \in G \ (g \cdot x = y)$$

E, F: equivalence relations on std. Borel spaces X, Y, resp.,

$E \leq_B F$, or E is Borel reducible to F, if there is a Borel function $f : X \to Y$ such that

$$xEx' \Leftrightarrow f(x)Ff(x')$$

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An equivalence relation E is called finite if each E-class is finite.

E is called countable if each E-class is countable.

- If G is a countable discrete group, then any E_G^X is countable.

Theorem (Feldman–Moore)

Any countable Borel equivalence relation is the orbit equivalence relation induced by some Borel action of some countable discrete group.

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An equivalence relation E is hyperfinite if $E = \bigcup E_n$ with each E_n a finite equivalence relation and each $E_n \subseteq E_{n+1}$.

Example

E_0 on $2^{\mathbb{N}}$:

$$xE_0y \Leftrightarrow \exists n \forall m \geq n \; (\, x(m) = y(m) \,)$$

Define

$$xE_{0,n}y \Leftrightarrow \forall m \geq n \; (\, x(m) = y(m) \,)$$

Each $E_{0,n}$ is finite, and E_0 is the increasing union of the $E_{0,n}$'s

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For a countable Borel equivalence relation E, E is hyperfinite iff $E \leq_B E_0$.

Theorem (Gao–Jackson)

For any countable abelian group G, E_G^X is hyperfinite.

So the countable case for abelian groups has been settled.

Conjecture

For any countable amenable group G, E_G^X is hyperfinite.

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For any countable amenable group G, E_G^X is hyperfinite.

An equivalence relation E is essentially countable if $E \leq_B F$ for some countable Borel equivalence relation F.

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If G is an abelian Polish group and E_G^X is essentially countable, then E_G^X is essentially hyperfinite.

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A topological group is non-archimedean if it has a nbhd base of its identity consisting of open (hence clopen) subgroups.

Theorem (Ding–Gao)

If G is a non-archimedean abelian Polish group and $E \leq_B E_G^X$ is essentially countable, then E is essentially hyperfinite.

- Locally compact Polish groups induce essentially countable orbit equivalence relations. (Kechris)

Corollary

If G is a locally compact non-archimedean abelian Polish group, then E_G^X is essentially hyperfinite

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If G is a locally compact non-archimedean abelian Polish group, then E_G^X is essentially hyperfinite

An equivalence relation E is smooth (or concretely classifiable) if $E \leq_B =_{\mathbb{R}}$.

E is hypersmooth if it is the increasing union of smooth equivalence relations.

- If E is both hypersmooth and essentially countable, then E is essentially hyperfinite. (Kechris–Louveau)

Theorem (C.)

If G is (Borel) isomorphic to the sum of a countable abelian group with a countable sum of copies of $\mathbb{R}$ and $\mathbb{T}$, then E_G^X is hypersmooth.

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Theorem (Pontryagin?)

Every locally compact abelian group is topologically isomorphic to $\mathbb{R}^n \times H$ for some non-negative integer n and abelian group H which has a compact open subgroup K.

- We can "mod out" compact subgroups, i.e., if G has a compact subgroup K, then $E_G^X \leq_B E_{G/K}^Y$ for a Borel $Y \subseteq X$. (Jackson–Kechris–Louveau)
- Polish groups are second countable, so then $\frac{\mathbb{R}^n \times H}{\{0\}^n \times K} \cong \mathbb{R}^n \times H/K$ where H/K is a *countable* abelian group.

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