Stone duality for infinitary logic

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$$\text{syntax} \leftrightsquigarrow \text{semantics}$$

Main examples

- propositional logic $\mathcal{L}_{\omega 0}$ (Stone duality)
- first-order logic $\mathcal{L}_{\omega\omega}$ (Makkai duality)
- infinitary first-order logic $\mathcal{L}_{\omega_1\omega}$

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Model of $\mathcal{T}$: $M : \mathcal{L} \to 2 = \{0, 1\}$ s.t. every $\phi \in \mathcal{T} \mapsto 1$ ($M \models \mathcal{T}$)
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Lindenbaum–Tarski algebra $\langle \mathcal{L} \mid \mathcal{T} \rangle := \{\mathcal{L}_{\omega 0}\text{-formulas}\}/\sim$ where

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$=$ Boolean algebra presented by generators $\mathcal{L}$, relations $\mathcal{T}$ $(= 1)$

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Hence,

$$\mathrm{Mod}(\mathcal{L}, \mathcal{T}) \cong \mathbf{Bool}(\langle \mathcal{L} \mid \mathcal{T} \rangle, 2)$$

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Stone duality

2 is a dualizing object, i.e., has two commuting structures:

- $2 \in \mathsf{Bool}$;
- $2 \in \mathsf{Top}$ (= topological spaces);
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Hence for $A \in \mathsf{Bool}$, $A^* := \mathsf{Bool}(A, 2) \subseteq 2^A \in \mathsf{Top}$;

$$A^{**} := \mathsf{Top}(A^*, 2) \subseteq 2^{A^*} \in \mathsf{Bool}$$

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$$\begin{aligned} \eta_A : A &\longrightarrow A^{**} := \mathsf{Top}(A^*, 2) \subseteq 2^{A^*} \in \mathsf{Bool} \\ a &\longmapsto (x \mapsto x(a)). \end{aligned}$$

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Theorem (Stone duality – algebraic)

For every $A \in \mathsf{Bool}$, $\eta_A : A \to A^{**}$ is an isomorphism.

Strong completeness for $\mathcal{L}_{\omega 0}$

When $A = \langle \mathcal{L} \mid \mathcal{T} \rangle$ for a propositional $\mathcal{L}_{\omega 0}$-theory $\mathcal{T}$, Stone duality becomes:

Theorem (Stone duality – logical)

For every $\mathcal{L}_{\omega 0}$-theory $\mathcal{T}$, we have an isomorphism

$$\begin{aligned} \eta_{\mathcal{T}} : \langle \mathcal{L} \mid \mathcal{T} \rangle &\longrightarrow \langle \mathcal{L} \mid \mathcal{T} \rangle^{**} \cong \mathrm{Mod}(\mathcal{L}, \mathcal{T})^{*} \cong \mathrm{Clopen}(\mathrm{Mod}(\mathcal{L}, \mathcal{T})) \\ [\phi] &\longmapsto \{ M \in \mathrm{Mod}(\mathcal{L}, \mathcal{T}) \mid M \models \phi \}. \end{aligned}$$

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- injectivity: for any two $\mathcal{L}_{\omega 0}$-formulas ϕ, ψ, if $\phi \Leftrightarrow \psi$ in all models of $\mathcal{T}$, then $\mathcal{T} \vdash \phi \Leftrightarrow \psi$ (completeness theorem)

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- surjectivity: any clopen set of models is named by an $\mathcal{L}_{\omega 0}$-formula (definability theorem)

Dual adjunctions

Stone duality is usually phrased in terms of a dual adjunction

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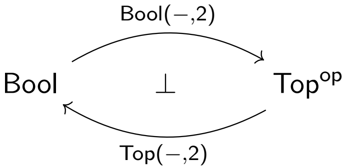

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By a general fact about adjunctions, this is equivalent to:

Theorem (Stone duality – algebraic, II)

The functor $\mathrm{Bool}(-, 2) : \mathrm{Bool} \to \mathrm{Top}^{\mathrm{op}}$ *is fully faithful, i.e., a bijection* $\mathrm{Bool}(A, B) \xrightarrow{\sim} \mathrm{Top}(B^*, A^*)$ *on each homset.*

Interpretations

An interpretation $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$ is a syntactic recipe for uniformly turning $M \in \mathrm{Mod}(\mathcal{L}', \mathcal{T}') \mapsto F^*(M) \in \mathrm{Mod}(\mathcal{L}, \mathcal{T})$.

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Formally, F is a Boolean homomorphism $\langle \mathcal{L} \mid \mathcal{T} \rangle \to \langle \mathcal{L}' \mid \mathcal{T}' \rangle$, i.e.,

- for each $P \in \mathcal{L}$, we have an $\mathcal{L}'_{\omega 0}$-formula $F(P)$;
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$$F^*(M) := M \circ F : \langle \mathcal{L} \mid \mathcal{T} \rangle \to 2.$$

In other words, $F^* = \mathrm{Bool}(F, 2) : \langle \mathcal{L}' \mid \mathcal{T}' \rangle^* \to \langle \mathcal{L} \mid \mathcal{T} \rangle^*$.

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Theorem (Stone duality – logical, II)

For any two propositional theories $(\mathcal{L}, \mathcal{T}), (\mathcal{L}', \mathcal{T}')$, every continuous map $\mathrm{Mod}(\mathcal{L}', \mathcal{T}') \to \mathrm{Mod}(\mathcal{L}, \mathcal{T})$ is induced by a unique interpretation $F : (\mathcal{L}, \mathcal{T}) \to (\mathcal{L}', \mathcal{T}')$.

The spatial side

Theorem (Stone duality – spatial)

Up to homeomorphism, spaces of the form A^* *for* $A \in \mathrm{Bool}$ *(i.e.,* $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ *for propositional theories* $(\mathcal{L}, \mathcal{T})$*) are exactly the compact Hausdorff zero-dimensional spaces (Stone spaces).*

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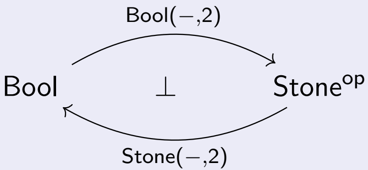
Corollary (Stone duality – complete)

We have a dual adjoint equivalence of categories



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$\mathrm{Mod}(\mathcal{L}, \mathcal{T})$:= category of (set-based) models

$$\mathcal{M} = (M,\ R^{\mathcal{M}} \subseteq M^n)_{n\text{-ary } R \in \mathcal{L}}$$

of $\mathcal{T}$ and elementary embeddings

Syntactic categories

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“presented” by generators $\mathcal{L}$ and relations $\mathcal{T}$, so that

$$\mathrm{Mod}(\mathcal{L}, \mathcal{T}) \cong \mathrm{BoolCoh}(\langle \mathcal{L} \mid \mathcal{T} \rangle, \mathrm{Set}) =: \langle \mathcal{L} \mid \mathcal{T} \rangle^*$$

$(\mathrm{BoolCoh}(\mathrm{A}, \mathrm{B}) :=$ category of Boolean coherent functors $\mathrm{A} \to \mathrm{B})$

Ultracategories and pretoposes

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Hence, for a Boolean coherent category A, $\mathsf{A}^* := \mathsf{BoolCoh}(\mathsf{A}, \mathsf{Set})$ is an ultracategory; and we have an evaluation functor

$$\eta_{\mathsf{A}} : \mathsf{A} \longrightarrow \mathsf{A}^{**} := \mathsf{Ultra}(\mathsf{BoolCoh}(\mathsf{A}, \mathsf{Set}), \mathsf{Set}) \in \mathsf{BoolCoh}.$$

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However, Set is a special Boolean coherent category, a pretopos:

- it has finite disjoint unions (well-behaved coproducts);
- it has quotients by equiv rels (well-behaved coequalizers).

Every $\mathsf{A} \in \mathsf{BoolCoh}$ has a pretopos completion $\overline{\mathsf{A}}$.

Makkai duality

Theorem (Makkai 1987)

For every $\mathsf{A} \in \mathsf{BoolCoh}$, $\eta_{\mathsf{A}} : \mathsf{A} \to \mathsf{A}^{**}$ *is the canonical embedding into its pretopos completion (i.e.,* $\mathsf{A}^{**} \cong \overline{\mathsf{A}}$*).*
In particular, if A *is already a pretopos, then* $\eta_{\mathsf{A}} : \mathsf{A} \cong \mathsf{A}^{**}$*.*

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- conservativity (inj on subobj lattices): for any two $\mathcal{L}_{\omega\omega}$-formulas α, β w/ same free vars, if $\alpha^{\mathcal{M}} = \beta^{\mathcal{M}}$ for all $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T})$, then $\mathcal{T} \vdash \alpha \Leftrightarrow \beta$ (completeness)

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- conservativity (inj on subobj lattices): for any two $\mathcal{L}_{\omega\omega}$-formulas α, β w/ same free vars, if $\alpha^\mathcal{M} = \beta^\mathcal{M}$ for all $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathcal{T})$, then $\mathcal{T} \vdash \alpha \Leftrightarrow \beta$ (completeness)
- full on subobjects (surj on subobj lattices): any assignment $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathcal{T}) \mapsto S_\mathcal{M} \subseteq M^n$ preserving elem embeddings and ultraproducts is $S_\mathcal{M} := \alpha^\mathcal{M}$ for some α (definability)

(cont'd)

Makkai duality

Theorem (Makkai 1987)

For every $\mathsf{A} \in \mathsf{BoolCoh}$, $\eta_{\mathsf{A}} : \mathsf{A} \to \mathsf{A}^{**}$ *is the canonical embedding into its pretopos completion (i.e.,* $\mathsf{A}^{**} \cong \overline{\mathsf{A}}$*).*
In particular, if A *is already a pretopos, then* $\eta_{\mathsf{A}} : \mathsf{A} \cong \mathsf{A}^{**}$*.*

For $\mathsf{A} = \langle \mathcal{L} \mid \mathcal{T} \rangle$, η_{A} is the functor

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(cont'd)

- essentially surjective: any $\mathcal{M} \in \mathsf{Mod}(\mathcal{L}, \mathcal{T}) \mapsto S_{\mathcal{M}} \in \mathsf{Set}$ functorial in $\mathcal{M}$ and preserving ultraproducts is defined by an imaginary sort $A \in \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}$ (strong definability)

Recall: $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}$ = completion of syntactic category $\langle \mathcal{L} \mid \mathcal{T} \rangle$ under finite disjoint unions and quotients by equiv rels.

So, an imaginary sort for $\mathcal{T}$ is a quotient of a finite disjoint union of formulas (definable sets) by a definable equiv rel.

Makkai duality

Makkai duality is given by a 2-categorical adjunction ($\eta_{\mathsf{A}} : \mathsf{A} \to \mathsf{A}^{**}$ is adjunction unit), hence may be restated as

Corollary

The 2-functor $\mathsf{A} \mapsto \mathsf{A}^* : \mathsf{BoolCoh} \to \mathsf{Ultra}^{\mathsf{op}}$ *is fully faithful.*
In other words, for any two first-order theories $(\mathcal{L}, \mathcal{T}), (\mathcal{L}', \mathcal{T}')$*, every ultraproduct-preserving functor* $\mathsf{Mod}(\mathcal{L}', \mathcal{T}') \to \mathsf{Mod}(\mathcal{L}, \mathcal{T})$ *is induced by an interpretation* $F : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle} \to \overline{\langle \mathcal{L}' \mid \mathcal{T}' \rangle}$.

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Unlike with Stone duality, the "spatial side" of Makkai duality seems to be open:

Question

Is there a nice characterization of the ultracategories of the form $\mathsf{Mod}(\mathcal{L}, \mathcal{T})$?

Other dualities

- Stone–Priestley duality for distributive lattices $\rightsquigarrow$ strong completeness for positive propositional logic
- Hofmann–Mislove–Stralka duality for Horn propositional logic
- Gabriel–Ulmer (1971) duality for Cartesian first-order logic
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Theorem (ess. Loomis–Sikorski)

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This gives a strong completeness theorem for *countable* $\mathcal{L}_{\omega_1 0}$-theories.

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Syntactic category $\langle \mathcal{L} \mid \mathcal{T} \rangle$ defined as for $\mathcal{L}_{\omega\omega}$ (formulas, definable functions), is the Boolean σ-coherent category:

- finite limits and some countable colimits, encoding $\wedge, \bigvee, \exists$,
- obeying all compatibility conditions that hold in Set,
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Every $\mathsf{A} \in \mathsf{Bool}\sigma\mathsf{Coh}$ has a σ-pretopos completion $\overline{\mathsf{A}}$

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$=$ "Borel $\cong$-equivariant assignments of a countable set to each $\mathcal{M} \in \mathrm{Mod}(\mathcal{L}, \mathcal{T})$"

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Duality for $\mathcal{L}_{\omega_1\omega}$

Theorem (C.)

For every countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$, the evaluation functor

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is the canonical embedding into the σ-pretopos completion $\overline{\langle \mathcal{L} \mid \mathcal{T} \rangle}$.

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Proof uses ideas from invariant DST and topos theory.

Duality for $\mathcal{L}_{\omega_1\omega}$

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For any two countable $\mathcal{L}_{\omega_1\omega}$-theories $(\mathcal{L}, \mathcal{T}), (\mathcal{L}', \mathcal{T}')$, every Borel functor $\mathrm{Mod}(\mathcal{L}', \mathcal{T}') \to \mathrm{Mod}(\mathcal{L}, \mathcal{T})$ *is induced by an $\mathcal{L}_{\omega_1\omega}$-interpretation* $F : \overline{\langle \mathcal{L} \mid \mathcal{T} \rangle} \to \overline{\langle \mathcal{L}', \mathcal{T}' \rangle}$.

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Theorem (C.)

Up to Borel equivalence, the standard Borel groupoids $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ are exactly the open non-Archimedean Polish groupoids.

Polish groupoid: internal groupoid in Pol (spaces of objects and morphisms are Polish spaces, groupoid operations are continuous)
Open: product of open sets of morphisms is open
Non-Archimedean: every identity morphism has a neighborhood basis of open subgroupoids

Work in progress

Theorem (C.)

Every open Polish groupoid is Borel equivalent to $\mathrm{Mod}(\mathcal{L}, \mathcal{T})$ *for some* $\mathcal{L}_{\omega_1\omega}$*-theory* $\mathcal{T}$ *in the continuous logic for metric structures.*

Remains to develop theory of syntactic categories and prove "algebraic" side of duality theorem for continuous $\mathcal{L}_{\omega_1\omega}$.

Thank you