Defining n Choose k Algebras to Generate $\mathbb{M}_m$ Representers

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Finite Lattice Representation Problem

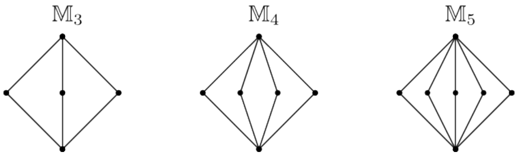
Open Problem

Is every finite lattice isomorphic to the congruence lattice of some finite algebra?

Palfy and Pudlak proved that the following are equivalent:

1. Any finite lattice is isomorphic to the congruence lattice of a finite algebra.
2. Any finite lattice is isomorphic to the congruence lattice of a finite, transitive G-set.

Our goal is to use finite transitive G-sets to represent the $\mathbb{M}_m$ lattices, e.g.



General Notation

- $\mathcal{A} = (A, G)$ denotes a transitive G-set.
- α, β, θ generally denote congruences of $\mathcal{A}$ which are not Δ or ∇
- If S is a set, $\mathcal{P}(S)$ denotes the power set, and $\mathcal{P}_k(S) = \{E \in \mathcal{P}(S) : |E| = k\}$.
- If X is a set and $Y \subseteq \mathcal{P}(X)$, then $X \times_{\in} Y = \{(x, y) \in X \times Y : x \in y\}$
- Δ and ∇ denote minimum and maximum congruences on an algebra, respectively.
- If $x \in A$, $\theta \in \text{Con}(\mathcal{A})$, $[x]_\theta = \{y \in A : (x, y) \in \theta\}$. Pronounced "$x$ mod θ"
- If $a \in A$, then $G_a = \{f \in G : f(a) = a\}$. Called the "stabilizer of a in G".

Dot Diagrams

Definition

If $\alpha, \beta \in \mathrm{Con}(\mathcal{A})$ with $\alpha \wedge \beta = \Delta$, then $\mathcal{A}$ can be represented as a **dot diagram (wrt α and β)** by plotting each $([x]_\alpha, [x]_\beta)$ for $x \in A$ as Cartesian coordinates.

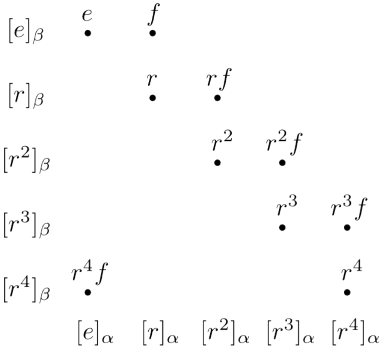
Each column of the dot diagram is an α class, and each row is a β class. Each $f \in G$ can be decomposed into a column permutation $f^{\mathcal{A}/\alpha}$ and a row permutation $f^{\mathcal{A}/\beta}$.

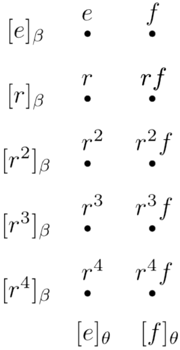
Example Dot Diagrams: Dihedral Group D_5

Different choices of $\alpha, \beta \in \text{Con}(\mathcal{A})$ can give different dot diagrams for the same algebra. Consider D_5 acting on itself on the left, an $\mathbb{M}_6$ representer.

- r is a rotation
- f is a reflection

- $\alpha = \Theta(e, r^4 f)$
- $\beta = \Theta(e, f)$
- $\theta = \Theta(e, r)$





Row Shape

Definition

For a row $[x]_\beta$, its **shape** $[[x]_\beta]_\alpha$ is the set of columns it intersects.

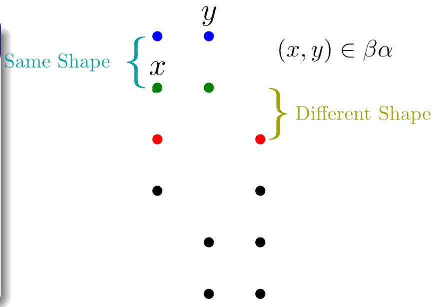
$$[[x]_\beta]_\alpha = \{[y]_\alpha : y \in [x]_\beta\}$$

Definition

$$(A/\beta)/\alpha = \{[[x]_\beta]_\alpha : x \in A\}$$

$$\beta\alpha = \ker(x \mapsto [[x]_\beta]_\alpha)$$

$\beta\alpha \in \mathrm{Con}(\mathcal{A})$ equates x and y if they are in rows with the same shape.



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Row shapes of $\mathbb{M}_m$ representers

Observation

$$\beta \subseteq \beta\alpha \subseteq \alpha \vee \beta$$

If $\mathrm{Con}(A, G) \cong \mathbb{M}_m$, then β is maximal, so there are 2 cases to consider:

Row shapes of $\mathbb{M}_m$ representers

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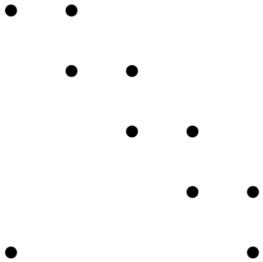
Figure: Case 1 D_5 Diagram



$\beta\alpha = \beta$, so each row has a unique shape.

Row shapes of $\mathbb{M}_m$ representers

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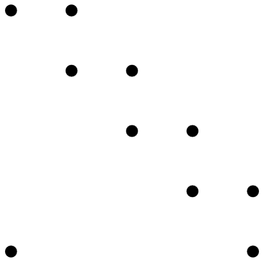
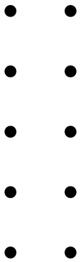


$\beta\alpha = \beta$, so each row has a unique shape.

Figure: Case 2 D_5 Diagram



$\beta\alpha = \nabla$, so all rows have the same shape, A/β. i.e. all rows intersect all columns.

Case 1: $\beta\alpha = \beta$

Each row can be identified with its shape, so $\mathcal{A}$ is embeddable in $(\mathcal{A}/\alpha) \times (\mathcal{A}/\beta)/\alpha$. The underlying set of this embedding is

$$(A/\alpha) \times_{\in} (A/\beta)/\alpha$$

Note that each $f \in G$ is fully determined by $f^{A/\alpha}$.

Definition

A transitive G-set $\mathcal{A}$ is an n **choose** k **algebra ($\binom{n}{k}$ algebra)** if there exists a transitive G-set $(\underline{A}, \underline{G})$ with $|\underline{A}| = n$ and a transitive G-set $(\overline{A}, \overline{G})$ with $\overline{A} \subseteq \mathcal{P}_k(\underline{A})$ such that there is a non-trivial subdirect embedding of $\mathcal{A}$ with underlying set $\underline{A} \times_{\in} \overline{A}$.

$\mathcal{A}$ is an $\binom{n}{k}$ algebra with $n = |A/\alpha|$ and $k = |A|/|A/\beta|$.

Case 2: $\beta\alpha = \nabla$

In case 2, each $f \in G$ requires both $f^{\mathcal{A}/\alpha}$ and $f^{\mathcal{A}/\beta}$ to be specified, so case 1 is preferred.

Question

When is Case 2 unavoidable? i.e. when is $\beta\alpha = \nabla$ for all distinct $\alpha, \beta \in \mathrm{Con}(\mathcal{A}) \setminus \{\Delta, \nabla\}$?

Case 2: $\beta\alpha = \nabla$

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Question

When is Case 2 unavoidable? i.e. when is $\beta\alpha = \nabla$ for all distinct $\alpha, \beta \in \mathrm{Con}(\mathcal{A}) \setminus \{\Delta, \nabla\}$?

Lemma

If $\mathrm{Con}(\mathcal{A}) \cong \mathbb{M}_m$ *for* $m \geq 3$ *and* $\beta\alpha = \nabla$ *for all distinct* $\alpha, \beta \in \mathrm{Con}(\mathcal{A}) \setminus \{\Delta, \nabla\}$, *then*

$$|A/\theta| = \sqrt{|A|}$$

for all $\theta \in \mathrm{Con}(\mathcal{A}) \setminus \{\Delta, \nabla\}$.

Thus, if every dot diagram representing $\mathcal{A}$ is a rectangle, then they must all be squares.

Classification of $\mathbb{M}_m$ representers

Theorem

If $\mathcal{A}$ is a transitive G-set and $\text{Con}(\mathcal{A}) \cong \mathbb{M}_m$ for $m \geq 3$, then one of the following holds:

1. For all $\theta \in \text{Con}(\mathcal{A}) \setminus \{\Delta, \nabla\}$, $|A/\theta| = \sqrt{|A|}$.
2. $\mathcal{A}$ is an n choose k algebra for some $n \geq 3$ and $2 \leq k \leq n-1$.

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Description of n Choose k algebras

In general, an n choose k algebra $\mathcal{A}$ will be identified with its subdirect embedding in $\underline{A} \times_{\in} \overline{A}$. G, $\underline{G}$, and $\overline{G}$ will be identified, as they are all different actions of the same group.

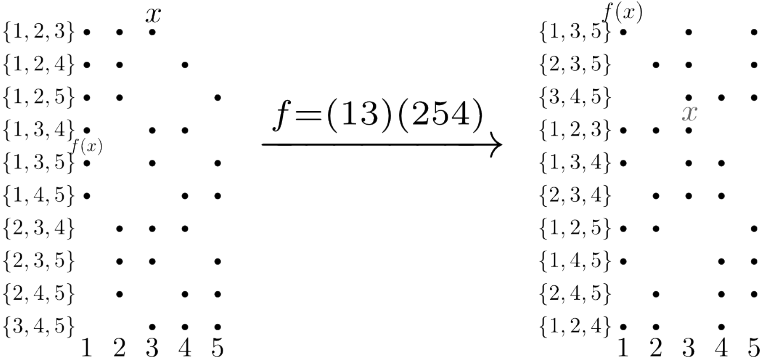
1. $\underline{A}$ is a club with n members.
2. $\overline{A}$ is a collection of committees with k members each.
3. A is the set of all teams that can be formed by choosing one member of a committee to be captain.
4. $x = (\underline{x}, \overline{x})$ denotes the team formed by choosing $\underline{x}$ to be the captain of committee $\overline{x}$. Define a similar projection $\tilde{x} := \overline{x} \setminus \{\underline{x}\}$
5. Each $f \in G$ is a permutation on the club, which acts pointwise on teams in A.

Example of an n choose k Algebra

Definition

The n choose k algebra $\mathcal{A}$ with $\overline{A} = \mathcal{P}_k(A)$ and $G = \mathrm{Sym}(\underline{A})$ is called the **symmetric n choose k algebra (Sym_k^n)**. If $G = \mathrm{Alt}(\underline{A})$ instead, then $\mathcal{A}$ is called the **alternating n choose k algebra (Alt_k^n)**.

Every $\binom{n}{k}$ algebra is a subreduct of the symmetric $\binom{n}{k}$ algebra.



Properties of n choose k algebras

The structure of an n choose k algebra depends only on the collection $\overline{A} \subseteq \mathcal{P}_k(\underline{A})$ and the group G acting transitively on $\underline{A}$.

Question

What are necessary and sufficient conditions on $\overline{A}$ and G defining an $\binom{n}{k}$ algebra?

Properties of n choose k algebras

The structure of an n choose k algebra depends only on the collection $\overline{A} \subseteq \mathcal{P}_k(\underline{A})$ and the group G acting transitively on $\underline{A}$.

Question

What are necessary and sufficient conditions on $\overline{A}$ and G defining an $\binom{n}{k}$ algebra?

Conditions on $\overline{A}$ and G

1. $\forall a, b \in \underline{A}$, $|\{E \in \overline{A} : a \in E\}| = |\{E \in \overline{A} : b \in E\}| \geq 2$
2. For each $E \in \overline{A}$, G_E acts transitively on E.
3. $\overline{A}$ is closed under G, and G acts transitively on $\overline{A}$.

General Methods for Obtaining n Choose k Algebras

Methods

1. Choose a transitive group G on n elements, then find a compatible $\overline{A} \subseteq \mathcal{P}_k(\underline{A})$.
2. Choose a G-set $\mathcal{A}$ and show that either $\mathcal{A}$ is an n choose k algebra or $\mathcal{A}/\theta$ is an n choose k algebra for some $\theta \in \mathrm{Con}(\mathcal{A})$.

A possible 3rd method is to choose some $\overline{A}$, then find a compatible group G. This will be the subject of future research.

Starting with G

Since $\overline{A}$ is closed under G and G acts transitively on $\overline{A}$, for any $E \in \overline{A}$,

$$\overline{A} = G(E)$$

Thus, it suffices to find a single $E \in \overline{A}$ such that conditions 1 and 2 are satisfied.

Theorem

Let $|\underline{A}| = n$, G be a group acting transitively on $\underline{A}$, and let $E \in \mathcal{P}_k(\underline{A})$ such that the following hold:

1. *G_E acts transitively on E*
2. *For all $a \in E$, $G_a \not\subseteq G_E$*

Then $(\underline{A} \times_{\in} G(E), G)$ is an n choose k algebra.

Choosing E to be the orbit of a permutation $f \in G$ ensures condition 1 is satisfied.

n choose k algebras as Factor Algebras

Theorem

Let $\mathcal{A}$ be a transitive G-set, let $\alpha, \beta \in \mathrm{Con}(A, G) \setminus \{\Delta, \nabla\}$ such that $\beta\alpha \neq \alpha \vee \beta$ and $\alpha \wedge \beta = \Delta$. Then $A/(\alpha \wedge \beta\alpha)$ is an n choose k algebra, where

$$n = |A/\alpha| \tag{1}$$

$$k = \frac{|A|}{|A/\beta|} \tag{2}$$

Corollary

If $\beta\alpha = \beta$ and $\alpha \wedge \beta = \Delta$, then $\mathcal{A}$ is an n choose k algebra

Modding out $\alpha \wedge \beta\alpha$ identifies rows with the same shape. If $\beta\alpha = \alpha \vee \beta$, then each column only intersects one row shape, so the factor algebra is not an n choose k algebra.

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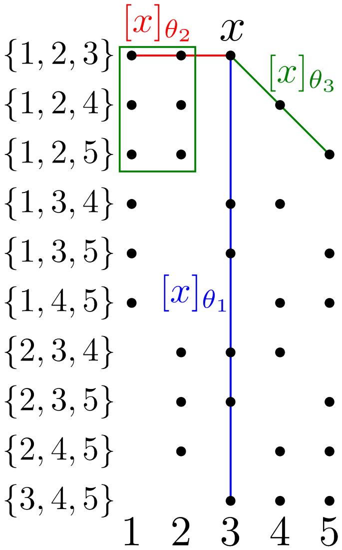
The maps $x \mapsto \underline{x}$, $x \mapsto \overline{x}$, and $x \mapsto \tilde{x}$ are homomorphisms, so their kernels are congruences.

Definition

Define congruences θ_1, θ_2, and θ_3 by

$$\theta_1 = \{(x, y) \in A^2 : \underline{x} = \underline{y}\}$$
$$\theta_2 = \{(x, y) \in A^2 : \overline{x} = \overline{y}\}$$
$$\theta_3 = \{(x, y) \in A^2 : \tilde{x} = \tilde{y}\}$$



Canonical Congruences, Part 2

Identifying $E \in \mathcal{P}(\underline{A})$ with E^c gives 3 additional congruences.

Definition

Define congruences ρ_1, ρ_2, and ρ_3 by

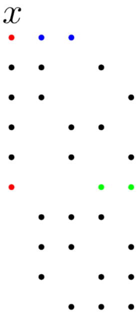
$$\begin{aligned}
\rho_1 &= \{(x,y) \in A^2 : (\tilde{x} = \overline{y}^c) \wedge (\tilde{y} = \overline{x}^c)\} \cup \Delta \\
\rho_2 &= \{(x,y) \in A^2 : \overline{x} \in \{\overline{y}, \overline{y}^c\}\} \\
\rho_3 &= \{(x,y) \in A^2 : \tilde{x} \in \{\tilde{y}, \tilde{y}^c\}\}
\end{aligned}$$

Note that $\rho_1 \subseteq \theta_1$, but $\theta_2 \subseteq \rho_2$ and $\theta_3 \subseteq \rho_3$.
These congruences are redundant unless certain cardinality conditions hold

General Case	Necessary Condition for Exception
$\rho_1 = \Delta$	$n = 2k - 1$
$\rho_2 = \theta_2$	$n = 2k$
$\rho_3 = \theta_3$	$n = 2k - 2$

ρ Congruence Dot Diagrams

ρ_1

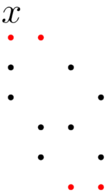


● $\in [x]_{\rho_1}$

● $\in \tilde{x}$

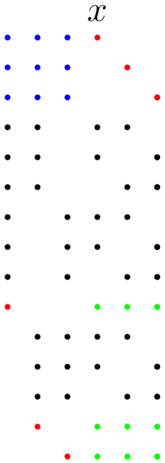
● $\in \overline{x}^c$

ρ_2



● $\in [x]_{\rho_2}$

ρ_3



● $\in [x]_{\rho_1}$

● $\in \tilde{x}$

● $\in \tilde{x}^c$

Congruence Lattices of Symmetric and Alternating n choose k Algebras

Theorem

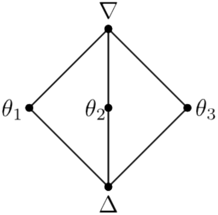
Let $n \geq 4$ and $2 \leq k \leq n-2$. If $n \notin \{2k, 2k-1, 2k-2\}$ then $\mathrm{Con}(\mathrm{Sym}^n_k) \cong \mathbb{M}_3$ with atoms $\theta_1, \theta_2, \theta_3$. The remaining cases all give isomorphic congruence lattices, described on the next slide.

Theorem

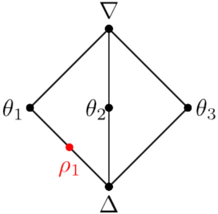
If $n \geq 4$ and $2 \leq k \leq n-2$, then $\mathrm{Con}(\mathrm{Alt}^n_k) = \mathrm{Con}(\mathrm{Sym}^n_k)$, except when $(n,k) = (4,2)$ or $(n,k) = (5,3)$.

Symmetric n choose k Congruence Lattices

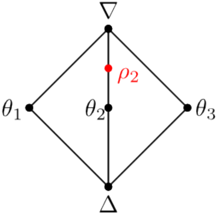
$n \notin \{2k, 2k-1, 2k-2\}$



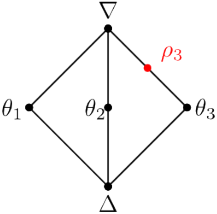
$n = 2k-1$



$n = 2k$



$n = 2k-2$



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Conclusion

Goal

Find transitive G-sets $\mathcal{A}$ such that $\text{Con}(\mathcal{A}) \cong \mathbb{M}_m$

Results

- Showed that all transitive G-set $\mathbb{M}_m$ representers are either n choose k algebras or have uniform congruence class size
- Developed methods of generating n choose k algebras.
- Examined congruences of n choose k algebras, especially the symmetric and alternating n choose k algebras.

Future Goals:

- Construct an n choose k algebra that represents a new $\mathbb{M}_m$.
- Develop method of generating n choose k algebras by starting with $\overline{A}$.
- Investigate $\mathbb{M}_m$ representers with uniform congruence class size.
- Consider generalizations of n choose k algebras.

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