Seven questions about ω^*

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BLAST, 2019

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The space $\beta\omega$

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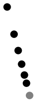

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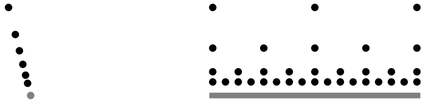

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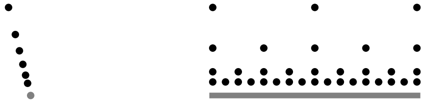

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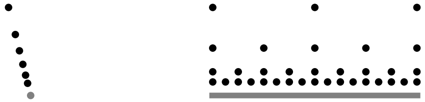

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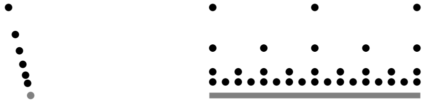

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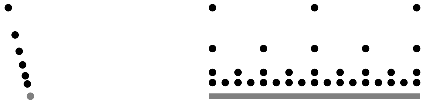

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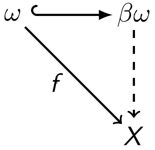
$\beta\omega$ is the "largest" compactification of ω:
i.e., if $\gamma\omega$ is any other compactification of ω, then there is a continuous surjection $\pi : \beta\omega \to \gamma\omega$ that fixes ω.

The space $\beta\omega$

$\beta\omega$ is the unique compactification of ω with the following *extension property*:

- every function mapping ω into a compact Hausdorff space extends continuously to $\beta\omega$.



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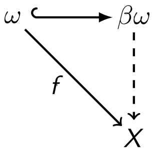

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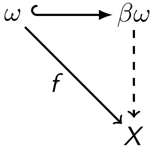

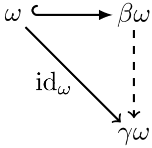

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The fact that $\beta\omega$ is the largest compactification of ω follows from the extension property.

The space ω^*

The space of all non-principal ultrafilters on ω, known as the *Stone-Čech remainder* of ω, is denoted

$$\omega^* = \beta\omega \setminus \omega.$$

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Theorem (Parovičenko, 1963)

Every compact Hausdorff space of weight $\leq \aleph_1$ is a continuous image of ω^*.

Corollary

Assuming the Continuum Hypothesis (CH), ω^* is a universal compact Hausdorff space of weight $\leq \aleph_1$; i.e., it is such a space, and it has every other such space as a continuous image.

Two classic questions about the topology of ω^*

Question 1: Efimov's problem

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- Assuming CH there is, for each $n \in \mathbb{N}$, a compact Hausdorff space having only n-dimensional infinite closed subspaces. Such spaces are Efimov, because converging sequences and ω^* are both 0-dimensional.
- How small can an Efimov space be? Recently a consistent example was found with weight $<$ the dominating number $\mathfrak{d}$.

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- It is known (without invoking any extra set-theoretic hypotheses) that $\omega^* \setminus \{u\}$ is non-normal for at least some $u \in \omega^*$.
- For example, this is true if u is a minimal point in the dynamical system $\sigma : \omega^* \to \omega^*$, where σ denotes the self-homeomorphism of ω^* known as the *shift map*.

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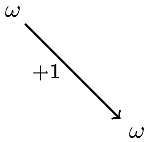

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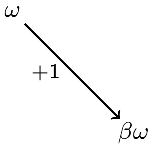

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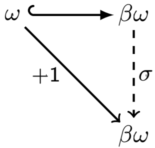

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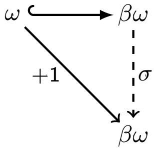

Equivalently, if $A \subseteq \omega$ then for every $u \in \beta\omega$ we have

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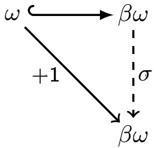

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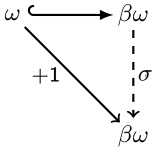

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The shift map restricts to a self-homeomorphism of ω^*.

- The shift map is the starting point for the entire theory of algebra in ω^*, which has far-reaching consequences in Ramsey theory and Diophantine approximation.

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If $f : X \to X$ and $g : Y \to Y$ are dynamical systems, then

- A *quotient map* (a.k.a. factor map) from f to g is a continuous function $\phi : X \to Y$ that sends the action of f on X to the action of g on Y, in the sense that $\phi \circ f = g \circ \phi$. In other words, the following diagram commutes.

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- If, furthermore, ϕ is a homeomorphism, then such a map is called an *isomorphism* from f to g.

Omega-limit sets

Given a dynamical system $f : X \to X$ and a point $x \in X$, the *omega-limit set* of x is the set of all limit points of the orbit of x:

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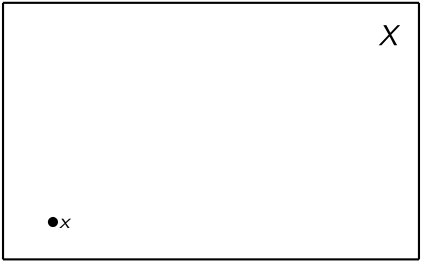

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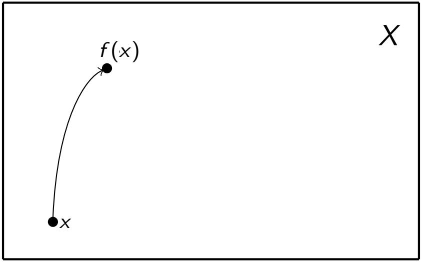

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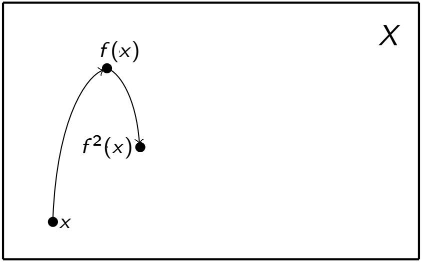

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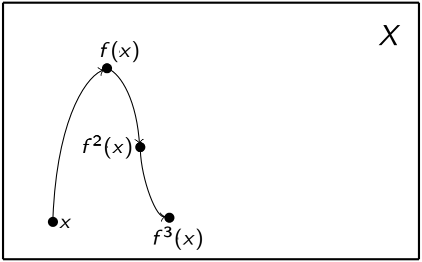

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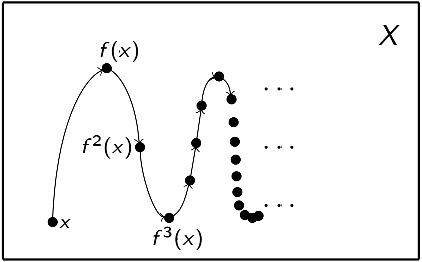

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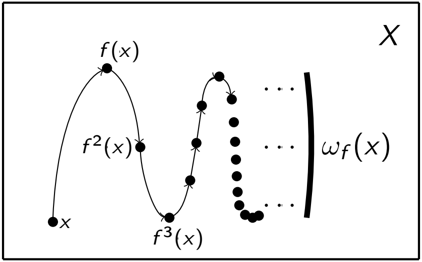

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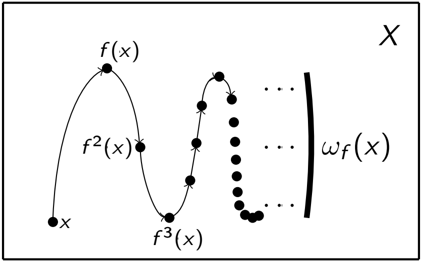

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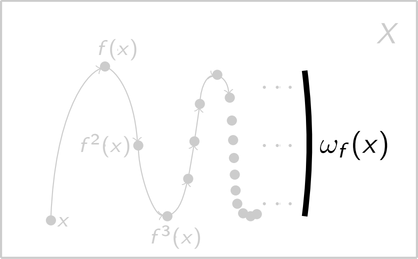

A universal property of the shift map

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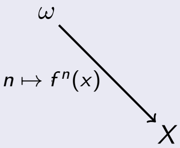

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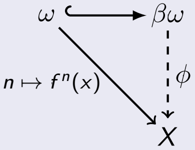

This function extends (uniquely) to a continuous function $\phi : \beta\omega \to X$. The restriction of ϕ to ω^* is a quotient mapping. □

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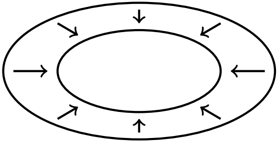

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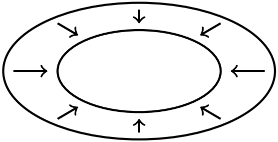

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If X is zero-dimensional (e.g., if it is the Stone space of some Boolean algebra), then this is equivalent to the condition $f(A) \not\subseteq A$ for every clopen $A \neq \emptyset, X$.

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- As mentioned on the previous slide, it follows from CH that σ and σ^{-1} are quotients of each other. (But this does not necessarily mean that they are isomorphic.) Assuming OCA + MA, σ and σ^{-1} are not even quotients of each other.
- A natural variation of van Douwen's question is whether CH implies already that σ and σ^{-1} are isomorphic.

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- This topology dualizes nicely to the space $\mathcal{A}ut(\mathcal{P}(\omega)/\mathrm{fin})$ of all automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$.
- Topologizing $\mathcal{H}(\omega^*)$ allows us to discuss formally what it means to have a "simple" or "complicated" collection of self-homeomorphisms of ω^*.

The quotients of the shift map, revisited

Observation

Assuming CH, *the set*

$$\mathcal{Q}_\sigma = \{h \in \mathcal{H}(\omega^*) \,:\, h \text{ is a quotient of } \sigma\}$$

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One may show (without too much effort) that $\mathcal{Q}_\sigma$ is not open. Thus CH makes $\mathcal{Q}_\sigma$ as simple as possible.

On the other hand . . .

Theorem (Brian, 2019)

Assuming OCA + MA, *the set*

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Thus while CH makes $\mathcal{Q}_\sigma$ as simple as possible, it seems that OCA + MA makes $\mathcal{Q}_\sigma$ very complicated.

A ZFC version

Theorem (Brian, 2019)

The isomorphism classes of σ and σ^{-1} cannot be separated by a Borel set in $\mathcal{H}(\omega^*)$. In particular, if σ and σ^{-1} are not isomorphic, then neither of their isomorphism classes is Borel.

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Corollary

If the isomorphism class of σ is Borel in $\mathcal{H}(\omega^*)$, then σ and σ^{-1} are isomorphic.

Two new questions

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- We know that $\mathfrak{c} \leq |\mathcal{H}(\omega^*)| \leq 2^{\mathfrak{c}}$. The number of isomorphism classes must be at least $\mathfrak{c}$, but a given class may have size $2^{\mathfrak{c}}$.

Metrizable reflections

One of the main tools used in both

- characterizing the weight $\leq\aleph_1$ quotients of σ, and
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is the notion of a *metrizable reflection*.

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If H denotes a (large fragment of) the set-theoretic universe, then a countable elementary submodel of H is a countable set M such that, for any formula in the language of set theory $\varphi(\vec{x})$, if $\vec{a} \in M$,

$$\varphi(\vec{a}) \text{ is true in } H \qquad \Leftrightarrow \qquad \varphi(\vec{a}) \text{ is true in } M$$

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Because M is elementary in H, the metrizable space X^M will share many important properties with the original X.

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Theorem (Noble and Ulmer, 1972)

If $x, y \in X$ and $\mathrm{Pr}(x) = \mathrm{Pr}(y)$, then $\mathrm{Pr}(h(x)) = \mathrm{Pr}(h(y))$.

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$$\begin{array}{ccc} X & \xrightarrow{\;\;h\;\;} & X \\ \mathrm{Pr}\downarrow & & \downarrow\mathrm{Pr} \\ X^M & \underset{h^M}{\dashrightarrow} & X^M \end{array}$$

This defines a map $h^M : X^M \to X^M$, the reflection of h in M.

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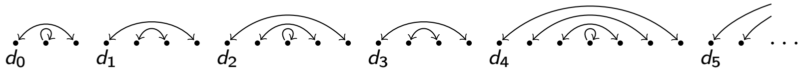

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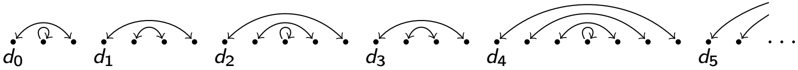
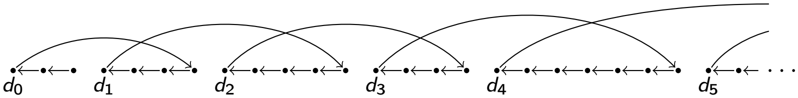


Then h^M is the desired isomorphism, because $h^{-1} \circ \sigma \circ h$ looks like this:



and by our choice of D, M cannot "see" where this map differs from σ^{-1}.

the Katowice problem

Question 6: the Katowice problem

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- If there were a homeomorphism $h : \omega^* \to \omega_1^*$, then the map $h \circ \sigma \circ h^{-1}$ would be topologically identical to the shift map.

the final question

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- If κ is any cardinal $> \omega_1$, then it can be proved that there is no weakly incompressible self-homeomorphism of κ^*.

The end

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Are there any *more* questions?