A non-pointfree approach to pointfree topology

Guram Bezhanishvili
New Mexico State University

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Outline

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Tutorial I: basics of pointfree topology

Tutorial II: basics of Priestley and Esakia dualities

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Tutorial II: basics of Priestley and Esakia dualities

Tutorial III: the study of frames through their spectra of prime filters

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The study of frames through their spectra of prime filters

Recap

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An Esakia space is extremally order disconnected if

$$U \text{ open upset} \quad \Rightarrow \quad \overline{U} \text{ open}$$

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EDEsa = the category of extremally order disconnected Esakia spaces and continuous order preserving maps $f : X \to Y$ satisfying

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for each open upset V of Y.

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Goal: Study frames by means of their extremally order disconnected Esakia spaces.

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Theorem: A prime filter x is completely prime iff ${\downarrow}x$ is clopen.

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Consequently, L is spatial.

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Theorem: The image of ε lands in $Y_{\mathcal{O}S}$ and $\varepsilon : S \to Y_{\mathcal{O}S}$ is the soberification of S.

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$$X_L = {\uparrow}\min X_L \text{ and } X_L = {\downarrow}\max X_L$$

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Recall: A frame L is regular if $a = \bigvee\{b \mid b \prec a\}$ for each $a \in L$.

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Theorem: L is regular iff R_a is dense in $\varphi(a)$ for each $a \in L$.

Corollary: The category **KRFrm** of compact regular frames is dually equivalent to the category of extremally order disconnected Esakia spaces satisfying

1. $\min X \subseteq Y$
2. The regular part of each clopen upset U is dense in U

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How can we realize such an equivalence?

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How can we realize such an equivalence?

Lemma: If L is a regular frame, then $Y_L \subseteq \min X_L$.

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Suppose $x \in Y_L$. Then $\downarrow x$ is clopen. If $x \notin \min X_L$, then there is $y < x$. Therefore, there is $a \in L$ with $x \in \varphi(a)$ and $y \notin \varphi(a)$. Let R_a be the regular part of $\varphi(a)$.

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Theorem: If L is compact regular, then $\max X_L$ is homeomorphic to the Gleason cover of Y_L.

Corollary: Gleason covers of compact Hausdorff spaces are realized as maximal spectra of their frames of opens.

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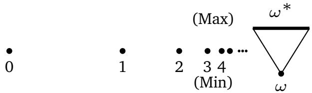

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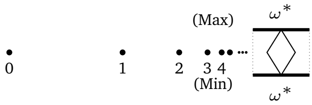

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Nuclei

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Such functions on L are called nuclei.

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Thus, sublocales are characterized by nuclei.

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The study of the tower

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is closely related to the Cantor-Bendixson analysis of derivative.

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Theorem: $N(L)$ is dually isomorphic to $N(X_L)$.

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From this we can derive the well-known results of Simmons and Isbell.

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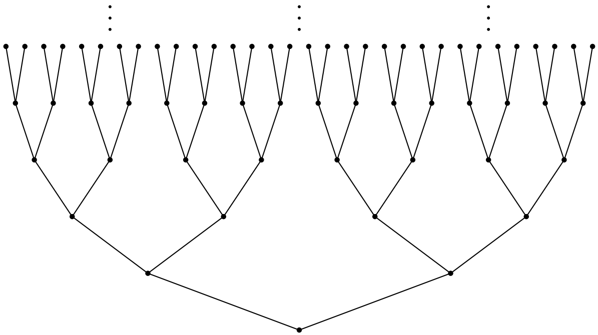

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Thank You!