A non-pointfree approach to pointfree topology

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Outline

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Tutorial II: basics of Priestley and Esakia dualities

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Tutorial II: basics of Priestley and Esakia dualities

Tutorial III: the study of frames through their spectra of prime filters

Tutorial II

Basics of Priestley and Esakia dualities

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One of the key properties in Stone's original definition was formulated in a more complicated manner than we formulate it now. The resulting class of spaces appeared quite exotic at the time, which could be one of the reasons for why Stone's duality for distributive lattices garnered less attention at the time than his duality for boolean algebras.

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While both Nerode and Hochster utilized the patch topology of a spectral topology, neither of them developed means to recover the original spectral topology from the patch topology.

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In 1975 Cornish recognized that the categories of spectral spaces and Priestley spaces are not only equivalent, but even isomorphic! We start by reviewing this isomorphism.

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The third defining property of a Priestley space is often referred to as the Priestley separation axiom.

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Consequently, the topology of a Priestley space is a Stone topology (i.e., a compact Hausdorff topology with a basis of clopen sets).

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Lemma. If $(X, \mathcal{T}, \leqslant)$ is a Priestley space, then $(X, \mathcal{T}^U)$ is a spectral space.

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Cornish (1975): **Spec** is isomorphic to **Pries**.

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Then $(X_L, \mathcal{O}_L)$ is the spectral space of L and $(X_L, \mathcal{T}_L, \subseteq_L)$ is the Priestley space of L.

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The join infinite distributive law yields that $\to$ is the residual of $\wedge$.

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Conversely, if L is a complete Heyting algebra, then

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Thus, L is a frame.

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But how should $\varphi(a) \to \varphi(b)$ be defined?

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For this to be well defined, we need the downset of clopen to be clopen.

Esakia spaces

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This demonstrates the non-symmetric feature of Heyting algebras and Esakia spaces.

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Since such spaces are order-analogues of extremally disconnected spaces, they are often called extremally order disconnected spaces.

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Pultr and Sichler (1988). A bounded distributive lattice L is a frame iff X_L is an extremally order disconnected Esakia space.

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We will see in Tutorial III that this point of view is very fruitful.

End of Tutorial II

Homomorphisms and nuclei

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Such functions on L are called nuclei.

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Thus, sublocales are characterized by nuclei.