A non-pointfree approach to pointfree topology

Guram Bezhanishvili
New Mexico State University

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Outline

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Tutorial I: basics of pointfree topology

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Tutorial II: basics of Priestley and Esakia dualities

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Tutorial III: the study of frames through their spectra of prime filters

Tutorial I

Basics of pointfree topology

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Starting from Isbell's 1972 paper Atomless parts of spaces pointfree topology became an independent branch of topology with its own internal problems.

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Some other closely related books are A compendium of continuous lattices (1980) and its new edition Continuous lattices and domains (2003), as well as a more recent Non-Hausdorff topology and domain theory (2013) by Goubault-Larrecq.

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Thus, the infinite distributive law $U \cap \bigcup_i V_i = \bigcup_i (U \cap U_i)$ holds

while the other infinite distributive law $U \cup \bigwedge_i V_i = \bigwedge_i (U \cap V_i)$ may fail.

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This defines a contravariant functor $\mathcal{O}$ from the category **Top** of topological spaces and continuous maps to **Frm**.

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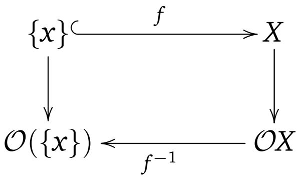

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Definition: A point of a frame L is a frame homomorphism $p : L \to \mathbf{2}$.

Completely prime filters

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Thus, points $\Leftrightarrow$ completely prime filters $\Leftrightarrow$ meet prime elements

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Thus, $\mathcal{O}(a \wedge b) = \mathcal{O}(a) \cap \mathcal{O}(b)$.

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Proof (continued).

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$p \in \mathcal{O}(\bigvee S)$

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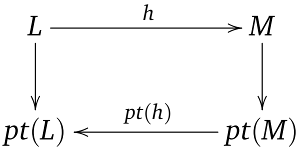

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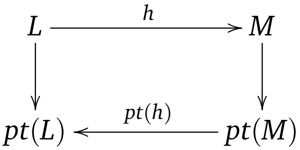

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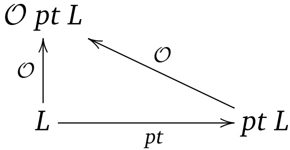

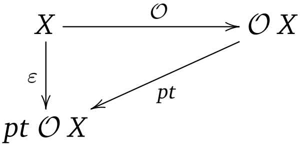

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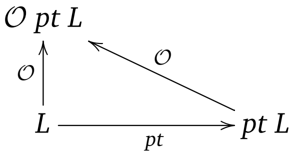

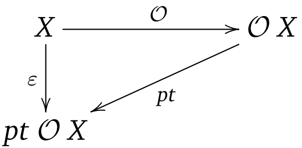

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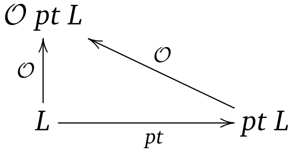

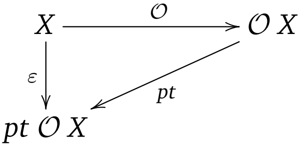

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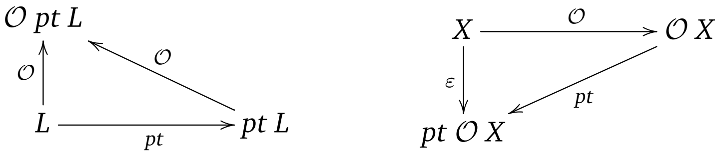

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Note. ε is continuous because $\varepsilon^{-1}\mathcal{O}(U) = U$ for all $U \in \Omega X$.

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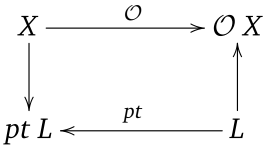

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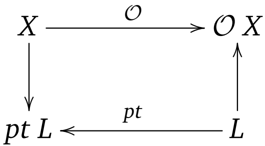

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Definition. Call X sober if ε is a bijection.

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$\varepsilon : X \to pt\ \mathcal{O}\ X$ is usually referred to as the soberification of X.

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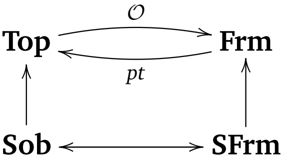

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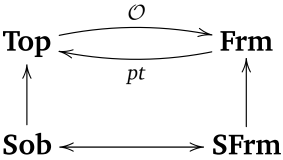

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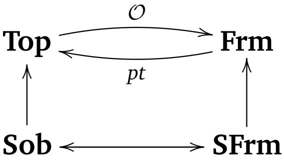

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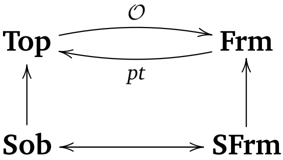

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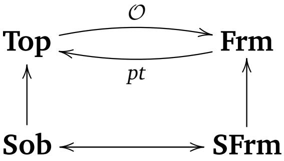

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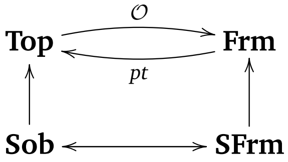

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It is customary in pointfree topology to replace **Top** with **Loc** and study **Loc** as the category of generalized spaces.

Compactness and local compactness pointfree

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Monteiro (1950s). A frame L is normal if for each $a, b \in L$, from $a \vee b = 1$ it follows that there are $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$.

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Lemma. A T_1-space X is completely regular iff the frame $\mathcal{O}X$ is completely regular.

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Every Hausdorff space is sober. (Basic idea: if X is Hausdorff, then the only irreducible closed sets are singletons.)

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Thus, **KHaus** is a subcategory of **Sob**. In fact, **KHaus** is a subcategory of **LKSob**.

Theorem. The dual equivalence of **Sob** and **SFrm** restricts to the dual equivalence of **KHaus** and **KRFrm**.

Summary

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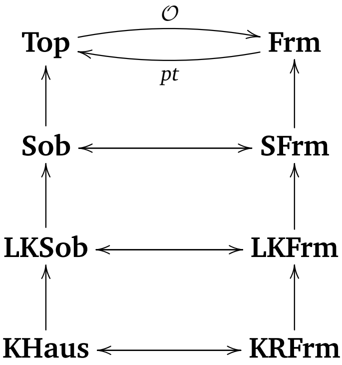

Non-spatial frames

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We will discuss a different possibility.

End of Tutorial 1

Homomorphisms and nuclei

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Such functions on L are called nuclei.

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Theorem. Frame homomorphisms are characterized by nuclei.

Thus, sublocales are characterized by nuclei.