Simple truncated archimedean vector lattices

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Truncs

Definition

A *truncated archimedean vector lattice*, or *trunc* for short, is an archimedean vector lattice with a *truncation*, i.e., a unary operation $G^+ \to G^+ = (g \mapsto \overline{g})$ satisfying the following for all $g, h \in G^+$.

- $g \wedge \overline{h} \leq \overline{g} \leq g$,
- $\overline{g} = 0$ implies $g = 0$,
- $ng = \overline{ng}$ for all n implies $g = 0$.✎

A *truncation homomorphism* is a vector lattice homomorphism $\theta \colon G \to H$ such that $\theta(\overline{g}) = \overline{\theta(g)}$. We denote the category of truncs and their homomorphisms by **T**.

Notation

- $\overline{ng/n}$ abbreviated to $g \wedge n$. The trunc has no element n.
- $g - \overline{g}$ abbreviated to $g \ominus 1$. The trunc has no element 1.
- $r(g/r \ominus 1)$ abbreviated to $g \ominus r$, $r \geq 0$. The trunc has no element r.✎

Prototypical truncs, the pointy ones

- $\mathcal{C}_0X \equiv \left\{ \tilde{g} \in \mathcal{C}X : \tilde{g}(*) = 0 \right\}$, where $(X, *)$ is a Tychonoff pointed space and $\mathcal{C}X$ is the family of continuous real valued functions on X. Here $\overline{\tilde{g}}(x) = \tilde{g}(x) \wedge 1$ for all $x \in X$. $\mathcal{C}_0X$ is a trunc which does not contain the constant function 1.
- $\mathcal{D}_0X \equiv \left\{ \tilde{g} \in \mathcal{D}X : \tilde{g}(*) = 0 \right\}$, where $(X, *)$ is a compact pointed space and $\mathcal{D}X$ is the family of continuous extended-real valued functions $\tilde{g}$ on X which vanish at the designated point and which are *almost finite*, i.e., $\tilde{g}^{-1}(\mathbb{R})$ is dense in X. Note that $\mathcal{D}_0X$ is not generally a trunc; we speak of *a trunc in $\mathcal{D}_0X$.*

Prototypical truncs, the pointfree ones

- $\mathcal{R}_0 L \equiv \{\, g \in \mathcal{R}L : g \text{ vanishes at } * \,\}$, where $(L, *)$ is a completely regular pointed frame and $\mathcal{R}L$ is the family of pointed frame maps $\mathcal{O}_* \mathbb{R} \to L$. Here

$$\overline{g}(-\infty, r) = \begin{cases} \top & \text{if } r > 1 \\ g(-\infty, r) & \text{if } r \leq 1 \end{cases}$$

  $\mathcal{R}_0 L$ is a trunc.

- $\mathcal{E}_0 q \equiv \{\, g \in \mathcal{R}_0 L : g \wedge n \text{ factors through } q \text{ for all } n \,\}$. Here $q : M \to L$ is a compactification. $\mathcal{E}_0 q$ is not generally a trunc; we speak of *a trunc in* $\mathcal{E}_0 q$.

The Yosida Representation for truncs

Theorem

- For any trunc G there is a unique compact Hausdorff pointed space $(X, *)$, a trunc $\widetilde{G}$ in $\mathcal{D}_0 X$, and a trunc isomorphism $\nu_G \colon G \to \widetilde{G} = (g \mapsto \tilde{g})$ such that $\widetilde{G}$ separates the points of X. The space X is called the *Yosida space of G*, designated $\mathcal{Y}_* G$.
- The representation is functorial. For every trunc homomorphism $\theta \colon G \to H$, where H is a trunk with Yosida space Y, there is a unique continuous pointed function k such that $\nu_H \circ \theta(g) = \nu_G(g) \circ k$, i.e., $\widetilde{\theta(g)} = \tilde{g} \circ k$.

$$
\begin{array}{ccc}
G & \xrightarrow{\nu_G} & \mathcal{D}_0 X \\
{\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_0 k} \\
H & \xrightarrow[\nu_H]{} & \mathcal{D}_0 Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{\tilde{g}} & \overline{\mathbb{R}} \\
{\scriptstyle k}\uparrow & \nearrow_{\widetilde{\theta(g)}} & \\
Y & &
\end{array}
$$

The Madden representation for truncs

Theorem

- Every trunc G is isomorphic to a subtrunc of $\mathcal{R}_0 L$ for some pointed frame L. L is called the *Madden frame of G*.
- This representation is functorial. For every trunc homomorphism $\theta\colon G \to H$, where H is a trunk with Madden frame M, there is a unique pointed frame map h such that $\mathcal{R}_0 h \circ \mu_G = \mu_H \circ \theta$, i.e., $\theta(g) = h \circ g$.

$$\begin{array}{ccc} G & \xrightarrow{\mu_G} & \mathcal{R}_0 L \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\mathcal{R}_0 h} \\ H & \xrightarrow[\mu_H]{} & \mathcal{R}_0 M \end{array} \qquad \begin{array}{ccc} L & \xleftarrow{g} & \mathcal{O}_*\mathbb{R} \\ {\scriptstyle h}\downarrow & \swarrow{\scriptstyle\theta(g)} & \\ M & & \end{array}$$

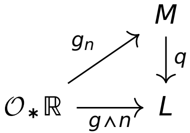
A hybrid representation theorem

A *compactification* is a dense pointed frame surjection $q\colon M \to L$ out of a compact regular pointed frame M.

Definition

For a compactification $q\colon M \to L$, let

$$\mathcal{E}_0 q \equiv \{\, g \in \mathcal{R}_0 L \mid \forall n \; (g \wedge n \text{ factors through } q) \,\}$$



$\mathcal{E}_0 q$ is closed under the scalar multiplication, the lattice operations, and truncation. It is not generally closed under addition or subtraction. However, a subset of $\mathcal{E}_0 q$ may be closed under all of the trunc operations. We speak of a *trunc in $\mathcal{E}_0 q$.*

A hybrid representation theorem

Theorem

Every trunc G is isomorphic to a trunc snugly embedded in $\mathcal{E}_0 q$ for a suitable compactification $q\colon M \to L$. This representation is functorial.

Unital components

Lemma

The following are equivalent for an element u of a trunc G.

- u is a *unital component of* G, i.e., $u \in \overline{G} = \{ \overline{g} \mid g \in G^+ \}$, and $u \wedge v$ is a component of v for each $v \in \overline{G}$. That is, $(u \wedge v) \wedge (v - u \wedge v) = 0$ for all $v \in \overline{G}$.
- $u = \overline{2u}$.
- $\tilde{u}$ is the characteristic function χ_U of some clopen subset $U \subseteq X$ which omits the designated point $*$ of X.

We denote the set of unital components of G by $\mathcal{UC}(G)$.

For any trunc G, $\mathcal{UC}(G)$ forms a *genralized Boolean algebra,* i.e., a distributive lattice with least element $\perp$ which admits relative complementation: for all a and b there exists c such that $c \vee b = a \vee b$ and $c \wedge b = \perp$.
$\mathcal{UC}(G)$ is Boolean, i.e., has a greatest element, iff G is unital, i.e., G contains an element $u \in G^+$ such that $\overline{g} = u \wedge g$ for all $g \in \overline{G}$. (This happens iff the designated point $* \in X$ is isolated.

Simple elements

Definition

An element g of a trunc G is *simple* if it is a linear combination of simple elements. A trunc G is called simple if all its elements are simple.

Theorem

The following categories are equivalent.

- The category **sT** of simple truncs with truncation homomorphisms.
- The category **gBa** of generalized Boolean algebras with morphisms which preserve the lattice operations and $\bot$.
- The category **iBa** of idealized Boolean algebras. The objects are of the form (B, I), where B is a Boolean algebra and I is a maximal ideal of B. The morphisms $f\colon (B, I) \to (C, J)$ are the Boolean homomorphisms $f\colon B \to C$ such that $f^{-1}(J) = I$.
- The category $\mathbf{zdK}_*$ of pointed Boolean spaces, i.e., zero dimensional compact Hausdorff pointed spaces.

Truncs bounded away from 0

Lemma

The following are equivalent for an element $g \geq 0$ in a trunc G.

- $\overline{ng} \in \mathcal{UC}(G)$ for some n.
- $u/n \leq \overline{g} \leq u$ for some $u \in \mathcal{UC}(G)$ and some n.
- There is a real number $\varepsilon > 0$ such that $\tilde{g}(x) > \varepsilon$ whenever $\tilde{g}(x) > 0$.
- There is a real number $\varepsilon > 0$ such that $\operatorname{coz} g = g(0, \infty) = g(\varepsilon, \infty)$.
- There is a real number $\varepsilon > 0$ such that $\operatorname{coz}(0, \varepsilon) = \bot$.

We say that g *is bounded away from* 0 if $|g|$ satisfies these conditions. We say that G *is bounded away from* 0 if every element of G is bounded away from 0.

The first characterization of simple truncs

Definition

An element $g \geq 0$ of a trunc G is said to be *bounded* if $g \leq n\overline{g}$ for some n. The *bounded part of* G is

$$G^* \equiv \{\, g \mid |g| \text{ is bounded} \,\},$$

a convex subtrunc of G. G is said to be *bounded* if $G = G^*$.

Theorem

The following are equivalent for a trunc G.

- G is simple.
- G is bounded and bounded away from 0.
- G is isomorphic to $\mathcal{LC}X$, the trunc of locally constant functions on a pointed Boolean space X which vanish at the designated point.

Hyperarchimedean truncs

Proposition

The following are equivalent for a trunc G.

- Every quotient of G by a convex ℓ-subgroup is archimedean.
- The spectrum of G is trivially ordered, i.e., every prime convex ℓ-subgroup is both maximal and minimal.
- Each principal convex ℓ-subgroup $G(g)$ is a cardinal summand, i.e., $G = G(g) \oplus g^{\perp}$ for all $g \in G^{+}$.

A trunc with these properties is called *hyperarchimedean.*

Example

Let X be the pointed Boolean space $(\omega + 1, \omega)$, and let $G \equiv \left\{ \tilde{a} + r\tilde{g}_0 \mid \tilde{a} \in \widetilde{A},\ r \in \mathbb{R} \right\}$, where

$$\widetilde{A} \equiv \{\, \tilde{a} \in \mathcal{D}_0 X \mid \operatorname{coz} \tilde{a} \text{ finite} \,\} \text{ and } \forall n\ \tilde{g}_0(n) = 1/n.$$

Then G is a hyperarchimedean trunc but it is not simple because it is not bounded away from 0.

A second characterization of simple truncs

Theorem

The following are equivalent for a trunc G.

- G is simple.
- G is hyperarchimedean and has enough unital components, i.e., for all $g \in G^+$ there exists $u \in \mathcal{UC}(G)$ such that $\overline{g} \leq u$.
- G is hyperarchimedean and bounded away from ∞, i.e., each $\tilde{g} \in \widetilde{G}$ vanishes on a neighborhood of the designated point $*$.

Thank you.