Abstracts

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The update of implicational bases for one-set extensions of a closure system

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The dynamic update of knowledge bases is a routine procedure in the realm of Artificial Intelligence. This application requires tractable representation, such as Horn logic or various versions of descriptive logic. The interest in Horn logic is easily explained by the fact that the reasoning in Horn logic is effective, while the reasoning in general propositional logic is intractable.

If some knowledge base is represented by a (definite) Horn formula Σ in variables $X = \{x_1, \dots, x_n\}$, then the set of its models $\mathcal{F}_\Sigma$ forms a lower subsemilattice in 2^X, which is often referred to as a *closure system* on X, or a *Moore family* on X [7]. Alternately, one can associate with Σ a *closure operator* φ on X, so that models from $\mathcal{F}_\Sigma$ are exactly the closed sets of φ. Also, Σ can be interpreted as a set of *implications* defining the closure operator φ. The general connections between Horn formulas (in propositional and first order logic), closure operators and their models were surveyed recently in [1].

The knowledge base requires an update if some of the models expire or the new models need to be incorporated into the existing base. In the current work we tackle the problem of re-writing the Horn formula Σ, when a new model A has to be added to the family $\mathcal{F}_\Sigma$. We will refer to it as the Singleton Horn Extension (SHE) Problem. If the knowledge base is given by a binary table, the associated closure operator is defined either on the set of columns or set of rows. Then SHE problem arises when adding new row or column to the table. The problem was earlier addressed in the general framework of closure systems in [8] and the framework of Horn-to-Horn belief revision in [4].

In our work we considered two special cases of the SHE problem: when formula Σ is given by the *canonical direct basis* of implications defining closure operator φ, and when it is given by its refined form of the *D-basis*. We will assume that one needs an algorithmic solution that provides at the output an updated formula $\Sigma^*(A)$ that is canonical direct, or, respectively, the D-basis of the extended closure system.

The *canonical direct basis* is well known in the literature and was introduced in multiple instances under various names. It was surveyed in [6] and shown to be a shortest *direct* basis of associated closure operator φ. A basis Σ is direct when the closure $\varphi(Y)$ of any set $Y \subseteq X$ is computed as follows:

$$\varphi(Y) = Y \cup \{d : (C \to d) \in \Sigma, C \subseteq Y\}.$$

The body-building formula $\Sigma(A)$ in [4] provides an update of the Horn formula Σ describing a closure system extended by a new closed set A, when the formula Σ represents the canonical direct basis of associated closure operator.

We observe that formula $\Sigma(A)$ is again the direct basis of the closure system extended by A, while it is not necessarily the canonical direct basis. We develop algorithmic solution for updated body-building formula $\Sigma^*(A)$ which represents the canonical direct basis.

The D-basis was introduced in [3] as a refined version of the canonical direct basis. In particular, it is a subset of the canonical direct basis and it is known to be *ordered direct*. Several algorithmic solutions were presented recently for extraction of the D-basis, when the closure system is given by some (definite) Horn formula Σ [9] or by a binary table [2].

We introduce the concept of a *binary-direct basis* of a closure system as an intermediate between direct and ordered direct bases. It turns out that the D-basis is the shortest binary-direct basis for a given closure system. This allows us to extend the approach used in the update of the canonical direct basis for the new family of bases, including the D-basis.

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Homogeneous universal and Atomic models: Boolean Algebras

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May 30, 2017

Abstract

I will discuss the evolution of Jónsson's construction of homogeneous-universal models, and in particular recent applications to infinitary logic, atomic model theory and abstract elementary classes.

I will survey three results. 1) The construction (with Koerwien and Laskowski) of an explicit complete $L_{\omega_1,\omega}$-sentence ϕ_n characterizing $\aleph_n$ for each n. This argument requires a new notion of n-dimensional amalgamation which allows the construction of *atomic* models in various uncountable cardinals. 2) These considerations led to constructions (with Souldatos) of complete $L_{\omega_1,\omega}$-sentences which have maximal models in more than one cardinal, but all below $\aleph_{\omega_1}$. Note that an abstract elementary class with amalgamation and joint embedding can have at most one maximal model. 3) Shelah and I have constructed (modulo presumably eliminable set theoretic hypotheses) a complete $L_{\omega_1,\omega}$-sentence with maximal models cofinally in the first measurable cardinal and forever if there is no measurable cardinal. Here Boolean algebra enters the picture as the crux is the construction of large atomic Boolean algebras, whose quotient modulo the finite joins of atoms, is atomless. The subtlety arises from considering when these quotients are actually free.

Equationally trivial algebras

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Abstract

An algebra is called equationally trivial if any system of (universally quantified) equations, which is satisfied by terms of the algebra, is satisfiable in every algebra. The development in last 10+ years fuelled by constraint satisfaction problems has brought several strong and surprising results about this general concept and variations thereof. I will talk about these results and discuss the open problems.

JÓNSSON'S CONTRIBUTIONS TO THE STUDY OF LATTICES OF LOGICS

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Among many contributions of Bjarni Jónsson to mathematics, two played an important role in the study of lattices of logics. One was his joint work with Alfred Tarski on canonical extensions of BAOs (Boolean algebras with operators). It yielded a relational representation of BAOs, which is instrumental in the study of completeness and canonicity in modal logic.

The other is his celebrated result on subdirectly irreducible algebras in congruence-distributive varieties. This opened the door for describing the upper parts of the lattices of modal and intermediate logics, which have a rather complicated structure.

The aim of this talk is to review these results, together with our current knowledge of the area, and list some of the remaining outstanding problems in the area, with the emphasis on the lattice of intermediate logics.

Representation of Convex Geometries by Circles on a Plane

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Abstract

Convex geometries are closure systems satisfying the anti-exchange axiom. Every convex geometry is represented by convex sets relative to some finite point configuration in n-dimensional space, for suitable n, by the result of K. Kashiwabara, M. Nakamura and Y. Okamoto (2005). In this work we show that not all finite convex geometries can be represented by finite configuration of circles on the plane. The question of the optimum space dimension for representation will be further discussed.

OPERATORS ON PAVELKA'S ALGEBRAS INDUCED BY FUZZY RELATIONS

MICHAL BOTUR

MV-algebras are algebraic models of Łukasiewicz multivalued propositional logic. Recall that MV-algebras are usually defined as algebras of type $\mathbf{A} = (A; \oplus, \neg, 0)$ such that

(MV1) $(A; \oplus, 0)$ is a commutative monoid,
(MV2) the double negation $\neg\neg x = x$ holds,
(MV3) the Łukasiewicz axiom $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ holds.

We recall that any MV-algebra $\mathbf{A} = (A; \oplus, \neg, 0)$ can be organized to bounded commutative residuated lattice satisfying divisibility, prelinearity and double negation law $(A; \vee, \wedge, \cdot, \rightarrow, 0, 1)$, where

$$
\begin{aligned}
x \vee y &:= \neg(\neg x \oplus y) \oplus y, \\
x \wedge y &:= \neg(\neg x \vee \neg y), \\
x \cdot y &:= \neg(\neg x \oplus \neg y), \\
x \rightarrow y &= \neg x \oplus y, \\
1 &:= \neg 0
\end{aligned}
$$

.

The Pavelka's logic enriches the Łukasiewicz's multi-valued logic with an infinite systems of constants belonging into a set $[0,1] \cap \mathbb{Q}$. These constants form a dense subalgebra of the *standard MV-algebra* defined as $([0,1]; \oplus, \neg, 0)$ where

$$
\begin{aligned}
x \oplus y &= \min\{x + y, 1\}, \\
\neg x &= 1 - x
\end{aligned}
$$

for all $x, y \in [0,1]$.

Definition 1. *By a* Pavelka's algebra *we mean an algebra* $\mathbf{A} = (A; \oplus, \neg, \{\mathbf{r} \mid r \in [0,1] \cap \mathbb{Q}\})$ *satisfying*

(P1) *the reduct* $(A; \oplus, \neg, \mathbf{0})$ *is an MV-algebra,*
(P2) *if* $r, s, t \in [0,1] \cap \mathbb{Q}$ *are such that* $r \oplus s = t$ *(computed in* $[0,1] \cap \mathbb{Q}$*) then* $\mathbf{r} \oplus \mathbf{s} = \mathbf{t}$ *(computed in* $\mathbf{A}$*),*
(P3) *if* $r, s \in [0,1] \cap \mathbb{Q}$ *are such that* $\neg r = s$ *(computed in* $[0,1] \cap \mathbb{Q}$*) then* $\neg \mathbf{r} = \mathbf{s}$ *(computed in* $\mathbf{A}$*).*

In the talk we will study special classes of Galois connections defined by:

Definition 2. *Let* $\mathbf{A} = (A; \oplus, \neg, \{\mathbf{r} \mid r \in [0,1] \cap \mathbb{Q}\})$ *and* $\mathbf{B} = (B; \oplus, \neg, \{\mathbf{r} \mid r \in [0,1] \cap \mathbb{Q}\})$ *be Pavelka's algebras. Let* $f \colon A \longrightarrow B$ *and* $g \colon B \longrightarrow A$ *be mappings forming a Galois connection between* A *and* B*. We say that* f *is a* strong left adjoint of g, *and we denote it by* $f \leftrightarrows g$, *if one of the following equivalent conditions holds*

(f) $\mathbf{r} \rightarrow f(x) = f(\mathbf{r} \rightarrow x)$ *for all* $x, \mathbf{r} \in A$,
(g) $\mathbf{r} \cdot g(x) = g(\mathbf{r} \cdot x)$ *for all* $x, \mathbf{r} \in A$.

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The main result of the talk is the following representation theorem.

Theorem 1. *i) If $R\colon I \times J \longrightarrow [0,1]$ is a fuzzy relation then mappings $f_R\colon [0,1]^I \longrightarrow [0,1]^J$ and $g_R\colon [0,1]^J \longrightarrow [0,1]^I$ defined by*

$$f_R(x)(j) = \bigwedge_{i\in I}(R(i,j) \to x(i)) \text{ for all } x \in A, j \in J, \tag{f_R}$$

$$g_R(x)(i) = \bigvee_{j\in J}(R(i,j) \cdot x(j)) \text{ for all } x \in B, i \in I, \tag{g_R}$$

satisfy $f_R \leftrightarrows g_R$.

ii) Let us have complete Pavelka's algebras $\mathbf{A} = (A; \oplus, \neg, \{\mathbf{r} \mid r \in [0,1] \cap \mathbb{Q}\})$ and $\mathbf{B} = (B; \oplus, \neg, \{\mathbf{r} \mid r \in [0,1] \cap \mathbb{Q}\})$ and mappings $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow A$ satisfying $f \leftrightarrows g$. Then there exists a fuzzy relation

$$R\colon \mathrm{Spec}_{\mathrm{M}}\,\mathbf{A} \times \mathrm{Spec}_{\mathrm{M}}\,\mathbf{B} \longrightarrow [0,1]$$

such that the diagram

$$\begin{tikzcd}
A \arrow[loop above, "gf"] \arrow[r, shift left, "f"] \arrow[d, hook, "n_{\mathbf{A}}"'] & B \arrow[l, shift left, "g"] \arrow[loop above, "fg"] \arrow[d, hook, "n_{\mathbf{B}}"] \\
{[0,1]^{\mathrm{Spec}_{\mathrm{M}}\,\mathbf{A}}} \arrow[loop below, "g_R f_R"] \arrow[r, shift left, "f_R"] & {[0,1]^{\mathrm{Spec}_{\mathrm{M}}\,\mathbf{B}}} \arrow[l, shift left, "g_R"] \arrow[loop below, "f_R g_R"]
\end{tikzcd}$$

commutes.

Finally, this theorem will be applied for a representation of Tense operators on Palvelka's algebras and monadic Pavelka's algebras.

Definition 3. *Let $\mathbf{A} = (A; \oplus, \neg, \{\mathbf{r} \mid r \in [0,1] \cap \mathbb{Q}\})$ be a Pavelka's algebra and let $G, H\colon A \longrightarrow A$ be operators such that for every $x, y \in A$,*

(PT1) $G(x \wedge y) = G(x) \wedge G(y)$ *and* $H(x \wedge y) = H(x) \wedge H(y)$,
(PT2) $\mathbf{r} \to G(x) = G(\mathbf{r} \to x)$ *and* $\mathbf{r} \to H(x) = H(\mathbf{r} \to x)$,
(PT3) $\neg H \neg G(x) \leq x$ *and* $\neg G \neg H(x) \leq x$.

Definition 4. *Let $\mathbf{A} = (A; \oplus, \neg, \{\mathbf{r} \mid r \in [0,1] \cap \mathbb{Q}\})$ be a Pavelka's algebra. Then by* monadic Pavelka's algebra *we mean a couple $(\mathbf{A}, \exists)$ where $\exists$ is a closure operator satisfying $\exists\neg\exists(x) = \neg\exists(x)$ and $\mathbf{r} \cdot \exists(x) = \exists(\mathbf{r} \cdot x)$.*

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On Groupoids of Relations with the Operation of C-intersection

This paper is dedicated to the memory of Bjarny Jónsson whose work inspired me to study algebras of relations

A set of binary relations closed with respect to some collection of operations on relations forms an algebra of relations. Any algebra of relations can be considered as partially ordered by the set-theoretic inclusion.

For any set Ω of operations on binary relations, denote by $R\{\Omega\}$ ($R\{\Omega, \subseteq\}$) the class of all algebras (partially ordered algebras) isomorphic to the ones whose elements are binary relations and whose operations are members of Ω. Let $Var\{\Omega\}$ ($Var\{\Omega, \subseteq\}$) be the variety and let $Qvar\{\Omega\}$ ($Qvar\{\Omega, \subseteq\}$) be the quasivariety generated by $R\{\Omega\}$ ($R\{\Omega, \subseteq\}$).

The following problems naturally arise when classes of algebras of relations are considered:

1) Find systems axioms for the class $R\{\Omega\}$ ($R\{\Omega, \subseteq\}$).

2) Find a basis of quasiidentities for the quasivariety $Qvar\{\Omega\}$ ($Qvar\{\Omega, \subseteq\}$).

3) Find a basis of identities for the variety $Var\{\Omega\}$ ($Var\{\Omega, \subseteq\}$).

4) Does the class $R\{\Omega\}$ ($R\{\Omega, \subseteq\}$) form a quasivariety?

5) Does the quasivariety $Qvar\{\Omega\}$ ($Qvar\{\Omega, \subseteq\}$) form a variety?

Alfred Tarski initiated the study algebras of relations from the point of view of universal algebra [1]. He considered algebras of relations with the following operations: Boolean operations $\cup, \cap, ^-$; operations of relational product $\circ$ and relational inverse $^{-1}$; constant operations $\emptyset$ (empty set), Δ (diagonal relation), U (universal relation). He showed that this class of algebras of relations is not a quasivariety, but the quasivariety generated by it forms a variety.

Bjarny Jónsson considered the class $R\{\circ, ^{-1}, \cap, \Delta\}$, proved that it forms a quasivariety, and posed the following question (see [2]): does this class form a variety? That paper by Bjarny Jónsson was the first research work that I read in my student years, and it determined my entire scientific career and was the source of my continued interest in the study of algebras of relations. I am very pleased that I was able to solve Jónsson's problem [3].

One of the most important classes of operations on relations is the class of *primitive-positive* operations [4] (in other terminology – Diophantine operations [5, 6]). An operation on relations is called primitive positive, if it can be defined by a formula containing in its prenex normal form only existential quantifiers and conjunctions. Note that all operations of Jónsson's algebras of relations are primitive-positive.

We will consider problems 1–5 for groupoids of relations, i.e., algebras of relation with one binary operation. The motivation for these investigations as well as some results can be found in [7, 8, 9]. Let as focus our attention on the following binary primitive-positive operation:

$$\rho * \sigma = \{(u, v) \in U \times U : (\exists w)(u, w) \in \rho \wedge (u, w) \in \sigma\},$$

where ρ and σ are relations on U.

Note that $\rho * \sigma = C(\rho \cap \sigma)$, where $C(\rho) = \{(u,v) \in U \times U : (\exists w)(u,w) \in \rho\}$ is the operation of cylindrification of relations [10]. For this reason, we call $\rho * \sigma$ the C-intersection of ρ and σ.

The main results are formulated in the following theorems. Their proofs are based on the description of equational and quasiequational theories of algebras of relations with primitive positive operations [5].

Theorem 1. The class $R\{*\}$ forms a variety. A groupoid $(A, \cdot)$ belongs to the class $R\{*\}$ if and only if it satisfies the identities:

$$(1)\ xy = yx,\quad (2)\ (xy)^2 = xy,\quad (3)\ (xy)y = xy,$$
$$(4)\ x^2y^2 = x^2y,\quad (5)\ (x^2y^2)z = x^2(y^2z).$$

Theorem 2. The class $R\{*, \subseteq\}$ forms a variety in the class of all partially ordered groupoids. A partially ordered groupoid $(A, \cdot, \leq)$ belongs to the class $R\{*, \subseteq\}$ if and only if it satisfies the identities (1)-(5) and the identities:

$$(6)\ x \leq x^2,\quad (7)\ xy \leq x^2.$$

Corollary. An algebra $(A, \cdot, \vee)$ of the type $(2,2)$ belongs to the variety $Var\{*, \cup\}$ if and only if $(A, \vee)$ is semilattice, and $(A, \cdot, \vee)$ satisfies the identities (1)-(5) and the identities:

$$(8)\ x(y \vee z) = xy \vee xz,\quad (9)\ x \vee x^2 = x^2,\quad (10)\ xy \vee x^2 = x^2.$$

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The theory of languages

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Abstract

We investigate the equational theory of algebras of formal languages with the constants empty language and unit language, the unary mirror image and Kleene star operators, and the binary operations of union, intersection and concatenation. We start by reducing the problem of deciding the validity of an equation over this signature to the equality of certain graph languages. This allows us to derive decidability, and to show that this problem is in fact ExpSpace-complete. Using this graph characterisation, and removing the Kleene star from our signature, we then propose a complete finite axiomatisation of this theory. This development was obtained using the proof assistant Coq.

We are interested in algebras of languages, equipped with the constants empty language (0), unit language (1, the language containing only the empty word), the binary operations of union ($+$), intersection ($\cap$), and concatenation ($\cdot$), and the unary operations of Kleene star ($\_^\star$) and mirror image, also called converse, ($\_^\smile$). We call these algebras *reversible Kleene lattices*. Given a finite set of variables X, and two terms e, f built from variables and the above operations, we say that the equation $e = f$ (respectively inequation $e \leq f$) is *valid* if the corresponding equality (resp. containment) holds universally. A *free representation* is a set $\mathcal{M}$ together with a map h from terms to elements of $\mathcal{M}$ such that $e = f$ is valid if and only if h maps e and f to the same element of $\mathcal{M}$.

It is well known that to any term over this syntax, one can associate a regular language, and that comparing regular languages is decidable. In fact, the problem of comparing regular expressions with intersection with respect to regular language equivalence is ExpSpace-complete [6]. The difference with the work presented here is that we are considering equations that are stable under substitution. Formally, that means that we do not interpret the letter a as the singleton language $\{a\}$, but rather as a universally quantified variable ranging over all languages. What is remarkable however is that testing the validity of equations in reversible Kleene lattices turns out to also be an ExpSpace-complete problem.

Fragments of this algebra have been studied:

Kleene algebra (KA, [5]). If we restrict ourselves to the operators of regular expressions (0, 1, $+$, $\cdot$, and $\_^\star$), then the free representation is the set of regular languages, with the usual definition of the language of an expression. Testing the validity of equations in KA is thus a PSpace-complete problem [7].

Kleene algebra with converse (KAC, [2]). If we add to KA the converse operation, then the free representation consists of regular expressions over a duplicated alphabet, with a letter a' denoting the converse of the letter a. The associated decision problem is still PSpace.

Identity-free Kleene lattices (KL$^-$, [1]). This algebra stems from the operators 0, $+$, $\cdot$, $\cap$ and $\_^+$, where the latter is the non-zero iteration. Andréka Mikulás and Németi studied this fragment, and showed that the free representation of this algebra consists of languages of series-parallel graphs, downward closed with respect to some graph preorder. We reformulated their results with Pous [4], and introduced a new class of automata, called Petri automata, able to recognise these languages of graphs. We provided a decision procedure to compare these automata, thus yielding an ExpSpace decision procedure for this theory. We prove ExpSpace-hardness by adapting a proof from [6].

The present work is then an extension of identity-free Kleene lattices, by adding unit and mirror image. The addition of mirror image is fairly simple, relying mainly on ideas from [2]. However, the seemingly small addition of 1 yields some complications. In fact, in [1, 4] there is a free representation of Kleene allegories, an algebra over the same signature as reversible Kleene lattices, but whose intended model is binary relations rather than languages. In that context, adding 1 means moving from *series-parallel* graphs to graphs of tree-width 2, that might have cycles. This is a significant problem for automata based decision procedures.

In the context of languages, adding 1 yields other problems. However, the free representation we get for reversible Kleene lattices remains more tractable than that of Kleene allegories. In particular we do not create cycles in series parallel graphs, but rather have to collect additional information. Let us illustrate this with the following inequation:

$$c \cdot (1 \cap a) \leq a \cdot c. \tag{1}$$

On the left hand side (LHS), the term $1 \cap a$ appears. This term is either equal to 1 if the empty word belongs to language a, or 0 otherwise. In the first case, the LHS is equal to c and we have $1 \leq a$, meaning that $c = 1 \cdot c \leq a \cdot c$. In the second case the LHS is equal to 0, which is contained in $a \cdot c$ as well. The key observation here is that the second case does not really matter: in a term build out of concatenations, intersections, converse, variables and units, if 0 appears somewhere then the term will always evaluate to 0 and thus be contained in any other term. The free representations we develop for union-free terms consist of pairs of a representation of a 1-free term and a set of language variables that are assumed to contain the empty word. This allows us to make the reasoning we used to study (1) automatic.

Following an approach similar to [4], we first construct the free representation of reversible Kleene lattices, and introducing a new Petri net-based automata model we show that testing the validity of equations is a decidable problem, and in fact an ExpSpace-complete one. This part of the development is available online [3]. Using the fact that without the Kleene star these free representations are finite, we build on ideas from [1] to find an prove correct a complete finite axiomatisation of the theory of languages over the signature $\langle 0, 1, +, \cdot, \cap, \_^{\smile} \rangle$, using the proof assistant Coq.

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The complexity of non-uniform CSPs

Andrei A. Bulatov

In a constraint satisfaction problem (CSP for short) the question is whether or not it is possible to find an assighment of values to given variables so that given constraints are satisfied. Various form of the CSP are ubiquitous in many areas of mathematics and computer science and have been extensively studied over the last several decades.

One of the prominent questions in the CSP research is the complexity of and algorithms for so called non-uniform CSPs, that is, CSPs in which the allowed values belong to a finite set D and the set of allowed constraints is also restricted. This set of allowed constraints is often called a constraint language, and the resulting restricted CSP for a constraint language Γ is denoted by $\mathrm{CSP}(\Gamma)$.

For every Γ the problem $\mathrm{CSP}(\Gamma)$ belongs to NP and many examples of constraint languages have been found that give rise to NP-complete problems or problems solvable in polynomial time. A complete complexity classification of non-uniform CSPs has been a major open problem since 1978, when Schaefer obtained such a classification for CSPs over a 2-element set. In 1993 Feder and Vardi proposed the CSP Dichotomy conjecture that states that for every constraint language Γ on a finite set $\mathrm{CSP}(\Gamma)$ is either solvable in polynomial time or is NP-complete. Since then the Dichotomy conjecture has attracted much attention of researchers from different fields.

The most fruitful method of approaching this conjecture relates constraint languages and the complexity of the CSP to properties of universal algebras. Many strong results in this area have therefore been obtained within the algebraic community. In this talk we review the main ideas that lead up to resolving the Dichotomy conjecture and outline the solution algorithm for $\mathrm{CSP}(\Gamma)$ that works in the cases when the problem can be solved in polynomial time.

The Borel complexity of embeddability between torsion-free abelian groups: a case for almost-fullness

Filippo Calderoni

Working in the framework of generalized descriptive set theory we prove that, for every uncountable cardinal $\kappa = \kappa^{<\kappa}$, there exists a Borel reduction from the relation of embeddability between κ-sized graphs to embeddability between κ-sized torsion-free abelian groups ([1]). The proof relies on a modification of an almost-full embedding of the category of graphs into the category of abelian groups, that was found by Przeździecki in [2].

As a corollary of our result we get that, for every uncountable cardinal $\kappa = \kappa^{<\kappa}$, the relation of embeddability between κ-sized torsion-free abelian groups is a complete $\mathbf{\Sigma}^1_1$ quasi-order.

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Dualizable algebras of arbitrary nilpotence class

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Abstract

In the early 2000s, Szabo and Quackenbush showed that groups and rings with a nonabelian nilpotent congruence are not dualizable. More recently, Bentz and Mayr showed that algebras in a modular variety that have a nonabelian supernilpotent congruence are also nondualizable. However, they did find an example of a dualizable algebra of nilpotence class 2 with infinite signature. We exhibit dualizable algebras of arbitrary nilpotence class with finite signature.

A Polynomial-Time Test for a Difference Term in an Idempotent Variety

William DeMeo, Ralph Freese, and Matthew Valeriote

Abstract. We consider the following practical question: given a finite algebra **A** in a finite language, can we efficiently decide whether the variety generated by **A** has a difference term? We answer this question in the idempotent case and then describe possible algorithms for constructing difference terms.

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Ramsey theory and the universal triangle-free graph

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Abstract

We present the solution to a long-standing open problem regarding the big Ramsey degrees of the universal triangle-free graph. The development of several new techniques are involved in the proof. One of these is a Ramsey theorem for strong coding trees, the proof of which uses the set-theoretic method of forcing to obtain a result in ZFC. These techniques will likely be useful for solving a collection of open problems regarding big Ramsey degrees of universal structures.

Ramsey theory on relational structures can be studied from two vantage points. Classically, structural Ramsey theory extends Ramsey's theorem to certain classes of finite relational structures. A Fraisse class of finite relational structures (such as finite ordered graphs) has the Ramsey property if for any structure A which embeds into a structure B, there is a structure C such that for any coloring of all copies of A in C into k colors, there is a copy of B in C in which all copies of A have the same color.

Of much recent interest, is the study of colorings of copies of a finite structure inside an infinite homogenous, universal structure. For example, it is well-known that any finite coloring of the vertices of the Rado graph can be reduced to one color on a subgraph which is also a Rado graph. For edges and other structures with more than one vertex, Sauer has proved this to be impossible. However, Sauer also proved that given a finite graph A, there is a number n(A) such that any coloring of all copies of A in the Rado graph into finitely many colors may be reduced to n(A) colors on a copy of the Rado graph. Using the terminology of Kechris, Pestov and Todorcevic, we say that the Rado graph has finite big Ramsey degrees. Similar results have been obtained for several other countable homogeneous structures, though most are still open.

The problem of finite big Ramsey degrees for the universal, homogeneous triangle-free graph H, constructed by Henson in 1971, has been open for some time, the problem being solved for vertex colorings by Komjath and Rodl in 1986, and for edge colorings by Sauer in 1998. The speaker has proved that for each finite triangle-free graph G, there is a number n(G) such that for each coloring of all copies of G in H into finitely many colors, there is a subgraph H' of H which is again universal triangle-free, and in which all copies of G in H' take on no more than n(G) many colors. The methods developed for this proof include new Ramsey theorems on trees, and will likely have bearing on solving the big Ramsey degree problem for other universal homogeneous relational structures omitting some type.

SYMMETRY IN QUOTIENTS OF PARTIALLY ORDERED SETS

DWIGHT DUFFUS

Given partially ordered sets X and Y, Y^X, the *(cardinal) power*, denotes the set of all order preserving maps of X to Y, with the pointwise order, that is, the order inherited from the $|X|$-fold power of Y. This operation is a familiar part of Birhkoff's generalized arithmetic based on sums, products and powers [1, 2] as well as his representation of distributive lattices as rings of sets. Aspects of the arithmetic have been studied for decades, with the most satisfying cancellation and refinement results for powers obtained by Jónsson and McKenzie (for instance, [9, 10, 13, 14]).

Quotients defined on powers play a role in investigations of symmetry properties of ordered sets. Given a partially ordered set P and any subgroup G of $\mathrm{Aut}(P)$, P/G denotes the set of orbits under G with the order induced by P: for $x, y \in P$, $[x] \leq [y]$ in P/G if there are $x' \in [x], y' \in [y]$ with $x' \leq y'$ in P. Stanley and others were interested in whether symmetry properties such as the unimodality and symmetry of level numbers and the strong Sperner property, which hold in Boolean lattice or power $\mathbf{2}^n$, are preserved by quotients (see [15, 16, 17]). They obtained several positive results but had no luck with symmetric chain decompositions (SCD's). Recall that a ranked partially ordered set P has an SCD or is a symmetric chain order (SCO) if it can be partitioned by chains, each saturated and symmetric about the middle of P.

The automorphism group of the Boolean lattice $\mathbf{2}^n$ is given by the natural action of the symmetric group S_n on the the subsets of the underlying n-set, say $[n]$. Several recent articles have dealt with the conjecture that for all subgroups G of S_n, $\mathbf{2}^n/G$ is an SCO (see [3] for the conjecture and [4, 5, 6, 7, 8, 11] for progress). In sum, we have made a bit of progress beyond the cyclic group $\mathbb{Z}_n$: see Theorem 1 in [6] and Theorem 1.1 in [4]. There are many interesting directions to go.

Problems for Specific Groups: Determine whether the following are SCO's.

(1) $\mathbf{2}^n/D_{2n}$ – This is a small step from $\mathbb{Z}_n$ but has not been solved (see [7]).

(2) $\mathbf{2}^n/A$ – Here A is an n-element abelian group and we take the regular representation of A in S_n. If n is prime, A is generated by a product of disjoint n-cycles so Theorem 1 in [6] applies, as it *may* for some embeddings for arbitrary n but rarely to the regular representation (see [6]).

(3) $\mathbf{2}^{\binom{n}{2}}/S_n^{\{2\}}$ – The set of all labelled graphs on vertex set $[n]$, ordered by containment, is isomorphic to $\mathbf{2}^{\binom{n}{2}}$. Let $S_n^{\{2\}}$ denote the action of S_n on the 2-subsets of $[n]$ and regard this as a subgroup of the full symmetric group on the set of all 2-subsets. This quotient is isomorphic to the set of unlabelled graphs on n vertices ordered by embedding. The analogous structure can be defined for k-uniform hypergraphs (see [15]).

(4) $\mathbf{2}^{\mathbf{k}\times\mathbf{t}}$ – Let $n = kt$ and think of $[n]$ as a set of t columns each of length k. Consider the permutations obtained by permuting within the columns independently and then permuting the columns. These form a subgroup of S_n, the wreath product $S_k \wr S_t$. The quotient $\mathbf{2}^{kt}/S_k \wr S_t$ is isomorphic to the

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power above, the set of order ideals of the chain product $\mathbf{k} \times \mathbf{t}$. So, Stanley's results verify rank symmetry and unimodality, and that the power has the strong Sperner property, but he was left to conjecture that it is an SCO [16]).

(5) $\mathbf{2}^{\mathbf{2}^n}$ – This is the free distributive lattice on n generators. This has been conjectured to be an SCO in [12]. An obvious question is whether it is a quotient.

There is a general question converse to the one in **(5)**: for which subgroups G of S_n is $\mathbf{2}^n/G$ a distributive lattice (see §**7** in [16])? Even more general questions involving powers and quotients could begin with Jónsson's result that under some conditions, $\mathrm{Aut}(X^Y) \cong \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ via the natural map. When the automorphism group is this nicely described, is there anything interesting to say about the quotient of X^Y by $\mathrm{Aut}(X^Y)$?

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Representing finite lattices as congruence lattices

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Abstract

We explore various aspects of the problem of representing a finite lattice as the congruence lattice of a finite algebra or an interval in the subgroup lattice of a finite group. We explore several constructions. Minimal representations are discussed including computer programs to find them. A catalog of representations of small lattices is given. It is shown that every lattice with at most seven elements, with only one possible exception, has a representation.

Undecidability for some varieties of commutative residuated lattices

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Residuated lattices were introduced in the study of rings via their ideal lattices, but also include as examples Boolean, Heyting and MV-algebras, lattice-ordered groups, and reducts of relation algebras, among others. They have also received a lot of attention as they serve as algebraic models of various substructural logics. A residuated lattice has a lattice reduct and a monoid reduct, and the monoid operation is residuated (via two division operations, functioning as implication in the logical interpretation). As can be seen in the above examples this additional monoid operation could be as varied as multiplication of ideals, the meet operation itself, group multiplication, or relational composition. The equational theory of residuated lattices is decidable, and this can be seen via the proof-theoretic calculus of the corresponding substructural logic, while their quasi-equational theory (actually their word problem) is undecidable, as one can embed semigroups. The same decidability and undecidability results hold for commutative residuated lattices (where the monoid operation is commutative). The addition of further identities changes things, however. All subvarieties of commutative residuated lattices axiomatized by non-trivial inequalities between monoid terms have a decidable quasi-equational theory (actually they enjoy the finite embeddability property). The same does not hold without the assumption of commutativity, as undecidability of the quasi-equational theory is preserved even in such subvarieties (with only a handful of exceptions). No proper subvariety of commutative residuated lattices that has undecidable quasi-equational theory was known, but we identify infinitely many such and among them some that have even undecidable quasi-equational theory. This is done by embedding a variant of Minski (counter/register) machines, and in particular one with undecidable halting problem. Our identities involve the join operation together with the monoid multiplication, and make use of join in order to model parallel computation. Also, since the addition of the identities affects computation a standard encoding used for the whole variety does not work, so we need to store the contents of the registers as powers of some fixed integer, whose size is determined by the identity that is being added. Identities that are captured by this process are described by the absence of positive solutions to some systems of linear inequalities. The correctness of the encoding of the machine is established using some relational semantics for residuated lattices based on Birkhoff polarities for lattices. The undecidability of the equational theory is then obtained through a reduction to it of the quasi-equational theorem via a deduction theorem.

Join-Completions and the Finite Embeddability Property

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Abstract

We present a systematic study of join-extensions and join-completions of ordered algebras, which naturally leads to a refined and simplified treatment of fundamental results and constructions in the theory of ordered structures ranging from properties of the Dedekind-MacNeille completion to the proof of the finite embeddability property for a number of varieties of ordered algebras.

Strange ultrafilters.

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Let F be a $\kappa-$complete ultrafilter over κ and $n, 0 < n < \omega$. How many ways are there to project F^n onto F?

Clearly, we have the projections to each of n many coordinates. But are there any other projections?
It is not hard to see that once F is normal, then - no.
Let us deal with general F's.

Our aim will be to show that:

Theorem 0.1 *1. If $n = 1$, then there is a unique way to project F to itself.*

2. For every $s, 1 < s < \omega$, it is possible to have F such that F^2 can be projected to F into $s-$many different ways.

3. Assuming strong, it is possible to have F such that F^2 can be projected to F into infinitely many ways.

Note that intuitively, if we have say three copies of F inside $F \times F$ at different places, then their envelope (the ultrafilter they generate) should be F^3. But F^3 is not Rudin - Kiesler below F^2.

The result has the following somewhat curious corollaries:

Corollary 0.2 *Let F be as in the previous theorem. Let P_F be the Prikry forcing with F and $\vec{\xi}$ a Prikry sequence. Then, in $V[\vec{\xi}]$ there is another Prikry sequence $\vec{\eta}$ for F(over V) which is disjoint from $\vec{\xi}$.*

Clearly the above situation is impossible once F is normal.

Corollary 0.3 *Let F be as in (3) of the previous theorem. Let P_F be the Prikry forcing with F and $\vec{\xi}$ a Prikry sequence. Then, in $V[\vec{\xi}]$ there are κ pairwise disjoint Prikry sequences $\langle \vec{\eta_\gamma} \mid 1 \leq \gamma \leq \kappa \rangle$ for F(over V) which are also disjoint from $\vec{\xi}$.*

Hypergraphs and cone lattices

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Abstract

Stated simply, "cone lattices" are structures which are (isomorphic to) the ideal completions of the incidence poset for a hypergraph. In this presentation, we motivate the term "cone lattice" and explore the structure of these objects. In particular we show that the following statements are logically equivalent.

1. A lattice $\mathcal{L}$ is order isomorphic to the frame of opens for a hypergraph endowed with the Classical topology.
2. A lattice $\mathcal{L}$ is bialgebraic, distributive, and its subposet of completely join-prime elements forms the incidence poset for a hypergraph.
3. A lattice $\mathcal{L}$ is a cone lattice.

In addition, we will show that well-known forbidden substructure conditions characterize those cone lattices associated with multigraphs and simple graphs. Time permitting, we will also indicate how this characterization leads to a Stone-type duality between the categories of hypergraphs coupled with anchored, finite-based (hyper) graph homomorphisms and cone lattices coupled with frame homomorphisms that preserve compact elements.

Algebraic foundations for qualitative calculi and networks.

Robin Hirsch, Marcel Jackson, and Tomasz Kowalski

Abstract

A qualitative representation ϕ is like an ordinary representation of a relation algebra, but instead of requiring $(a;b)^\phi = a^\phi|b^\phi$, as we do for ordinary representations, we only require that $c^\phi \supseteq a^\phi|b^\phi \iff c \geq a;b$, for each c in the algebra. A constraint network is qualitatively satisfiable if its nodes can be mapped to elements of a qualitative representation, preserving the constraints. If a constraint network is satisfiable then it is clearly qualitatively satisfiable, but the converse can fail. However, for a wide range of relation algebras including the point algebra, the Allen Interval Algebra, RCC8 and many others, a network is satisfiable if and only if it is qualitatively satisfiable.

Unlike ordinary composition, the weak composition arising from qualitative representations need not be associative, so we can generalise by considering network satisfaction problems over non-associative algebras. We prove that computationally, qualitative representations have many advantages over ordinary representations: whereas many finite relation algebras have only infinite representations, every finite qualitatively representable algebra has a finite qualitative representation; the representability problem for (the atom structures of) finite non-associative algebras is **NP-complete**; the network satisfaction problem over a finite qualitatively representable algebra is always in **NP**; the validity of equations over qualitative representations is **co-NP-complete**. On the other hand we prove that there is no finite axiomatisation of the class of qualitatively representable algebras.

A LATTICE THEORETIC APPROACH TO PERMUTOHEDRA

TRISTAN HOLMES

ABSTRACT. We present an introduction to generalized permutohedra suitable for an audience familiar with the basics of lattice theory. We begin by stating the definition of the permutohedron arising from a partially ordered set E and illustrate standard examples such as the weak Bruhat order on symmetric groups. There are limits to this construction that are overcome by Santocanale and Wehrung's generalized permutohedron. We conclude by stating several open questions.

Let E be a partially ordered set, and let $\delta_E := \{(p,q) \in E \times E \mid p < q\}$ be the *strict ordering* on E. We define $\mathrm{cl}(\mathbf{a})$ to be the transitive closure of any $\mathbf{a} \subseteq \delta_E$, and $\mathrm{int}(\mathbf{a}) = \delta_E \setminus \mathrm{cl}(\delta_E \setminus \mathbf{a})$. Set

$$\mathtt{P}(E) := \{\mathbf{a} \in \delta_E \mid \mathbf{a} = \mathrm{cl}(\mathbf{a}) = \mathrm{int}(\mathbf{a})\},$$
$$\mathtt{R}(E) := \{\mathbf{a} \in \delta_E \mid \mathbf{a} = \mathrm{cl}\ \mathrm{int}(\mathbf{a})\}.$$

We call $\mathtt{P}(E)$ the *permutohedron on E*, $\mathtt{R}(E)$ the *extended permutohedron on E* and order both by set containment.

Denote by $[n]$the chain consisting of n elements. For convenience, we say $\mathtt{P}(n) := \mathtt{P}([n])$. In this case $P(n)$ is isomorphic to the symmetric group on n letters equipped with the weak Bruhat order. More generally, modifications of this approach allow for a realization Reading's *Cambrian lattices of type A* by way of this closure operator. An important class of such lattices is the so called *Tamari lattices* $\mathtt{A}(n)$.

Theorem 1 (Pouzet et. al.)**.** *We call E* square free *if it contains no copy of the four element Boolean lattice. The partially ordered set* $\mathtt{P}(E)$ *is a lattice if and only if E is square free.*

Theorem 2 (Santocanale and Wehrung)**.** *The partially ordered set* $\mathtt{R}(E)$ *is an ortholattice for any partially ordered set E. Moreover,* $\mathtt{R}(E)$ *is the Dedekind-MacNielle completion of* $\mathtt{P}(E)$.

The weak Bruhat order has been studied from a combinatorial perspective for some time. A lattice theoretic approach to this and its possible generalizations is quite recent. A great deal of progress in this

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regard has been made in the last five years, summarized in the findings of Santocanale and Wehrung's latest paper on the topic.

Theorem 3 (Santocanale and Wehrung)**.**

A: *The equational theory of all* $\mathtt{P}(n)$ *and the equational theory of all* $\mathtt{A}(n)$ *are both decidable.*

B: *There exists a lattice identity that holds in all* $\mathtt{P}(n)$ *and fails in a certain 3,388-element lattice.*

C: *The equational theory of all extended permutohedra over arbitrary and possibly infinite partially ordered sets is trivial.*

There are numerous open questions relating to these results that can motivate further research.

- It is known that every Coxeter lattice of type B can be embedded into some $\mathtt{P}(n)$. Is this true for all Coxeter lattices? Are they at least members of the variety generated by all $\mathtt{P}(n)$?
- Is it decidable whether the class of lattices satisfying a given identity is contained in the variety generated by all permutohedra (but not extended permutohedra)? Can the variety of all permutohedra be defined by finitely many (and therefore one) lattice identity?
- Can the known results be extended to varieties and quasivarieties of permutohedra viewed as ortholattices, i.e., lattices with an additional unary operation of complementation?
- While it is known that the equational theory of all $\mathtt{P}(n)$ and of all $\mathtt{A}(n)$ are decidable, the known algorithms are intractable. Can tractable algorithms for these problems be found?

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On the enduring impact of Bjarni's work in Lattice Theory, Relation Algebras and Boolean Algebras with Operators

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Abstract: This tutorial will cover some of the highlights of Bjarni Jónsson's influential research on lattices, algebras of binary relations and the general theory of Boolean algebras with operators. In the first session we will consider some of the details of this work and how it lead to the modern view of algebraic logic, as well as several other research directions. In the second session we focus on how Bjarni's approach generalizes to residuated lattices, (topological) residuated frames and ordered partial algebras with complex algebras that are residuated Heyting algebras. Some applications to computer science and the semantics of concurrent programs will also be featured.

Listed below are some of Bjarni's papers that are directly related to this tutorial.

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Cologic for profinite structures and coalgebras

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Abstract

The existence of a categorical dual to first-order logic is hinted at in several independent bodies of work: (1) Projective Fraïssé theory, (2) The "cologic" of profinite groups (e.g. Galois groups), which plays an important role in the model theory of PAC fields, (3) Ultracoproducts and coelementary classes of compact Hausdorff spaces, (4) Universal coalgebras and coalgebraic logic.

In this talk, I will show how to generalize the role of the category of sets in ordinary first-order logic, replacing it with an arbitrary locally finitely presentable (LFP) category. Objects of this category are the domains of structures, while the full subcategory of finitely presentable objects is the category of variable contexts. Then, noting that categories of profinite structures (e.g. Stone, $\mathsf{ProFinGrp}$, etc.) are co-LFP, we can build this logic for their dual categories. The result is a first-order cologic, suitable for describing profinite structures and coalgebras via their finite quotients.

Using Gabriel-Ulmer duality, cologic (and any first-order logic on an LFP category) is interpretable in an ordinary multi-sorted first-order framework, which allows us to easily import theorems and notions from ordinary first-order model theory.

ON HORIZONTAL SUMS OF MV-ALGEBRAS

JAN KÜHR

Effect algebras are an abstraction of the additive structure of Hilbert space effects [3]; they are certain partially ordered partial commutative monoids. Lattice-ordered effect algebras can in a natural way be made into (universal) algebras $\langle A, \oplus, ', 0, 1\rangle$ and the class of such algebras, $\mathcal{E}$, is a variety [1]. Perhaps the best known subvarieties of $\mathcal{E}$ are the variety of MV-algebras, $\mathcal{MV}$, and the variety (term equivalent to the variety) of orthomodular lattices, $\mathcal{OM}$. I will focus on some common properties of the subvarieties of $\mathcal{E}$ that we studied in our earlier papers [2, 4, 5].

(i) The subdirectly irreducible members of these subvarieties, except for $\mathcal{MV} \vee \mathcal{OM}$, are either MV-chains or horizontal sums of MV-chains. I will characterize the class $\mathsf{Hor}(\mathcal{MV})$ of horizontal sums of MV-algebras and axiomatize the variety generated by $\mathsf{Hor}(\mathcal{MV})$; this is done using the "commutator" $\gamma(x, y) = (x' \oplus (x \oplus y)) \wedge (y' \oplus (y \oplus x))$. The class $\mathsf{Hor}(\mathcal{MV}_{\mathcal{C}})$ of horizontal sums of MV-chains generates a smaller variety.

(ii) The subvarieties in question, except for $\mathcal{MV} \vee \mathcal{OM}$, have "nice" ideals (that is, the congruence kernels are exactly the order ideals closed under $\oplus$). The class of all lattice effect algebras with such "nice" ideals, $\mathcal{N}$, is not closed under products. I will prove that every subvariety of $\mathcal{N}$ is contained, for some positive integer k, in the variety $\mathcal{N}_k = \mathcal{MV} \vee \mathcal{E}_k$ where $\mathcal{E}_k$ is the variety generated by horizontal sums of MV-chains of length $\leq k$. Relative to $\mathcal{E}$, the variety $\mathcal{N}_k$ is axiomatized by the identity $x \leq kx \oplus y$. Further, I will prove that the variety generated by $\mathcal{N}$ is equal to: (1) the variety generated by $\mathsf{Hor}(\mathcal{MV}_{\mathcal{C}})$, (2) the join of the $\mathcal{E}_k$'s, (3) the join of the $\mathcal{N}_k$'s.

(iii) Roughly speaking, certain elements of algebras in the varieties in question, including $\mathcal{MV} \vee \mathcal{OM}$, have special properties. For instance, the variety generated by $\mathsf{Hor}(\mathcal{MV})$ can be specified by the condition that the elements $\gamma(x, y)$ are central. Using this observation, I will axiomatize the varietal joins $\mathcal{N}_k \vee \mathcal{OM}$.

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Generalized Residuated Frames

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This talk reports about ongoing work in which we generalize the framework of residuated frames, introduced in [4] to give a semantic proof of cut admissibility for various axiomatic extensions of the basic Lambek calculus, and applied to the proof of the finite embeddability property and finite model property for some of these.

Our generalization concerns two aspects:

1. from the signature of residuated lattices to arbitrary normal lattice expansions; in particular, arbitrary signatures do not need to be closed under the residuals of each connective.
2. from structural rules of so-called simple shape to the more general class of analytic structural rules (cf. [5], Definition 4) in any signature of normal lattice expansions.

Specifically, for every signature $(\mathcal{F},\mathcal{G})$ [5] of normal lattice expansions (cf. [3][2]) we define the associated notion of $(\mathcal{F},\mathcal{G})$-frame, and we prove that

1. the cut rule is admissible in the Gentzen calculus associated with the basic normal lattice logic in the signature $(\mathcal{F},\mathcal{G})$. We prove this result by suitably generalizing the semantic argument given in the proof of Theorem 3.2 in [4].
2. the cut admissibility above transfers to extensions of the Gentzen calculus above with structural rules generalizing the notion of simple structural rules (cf. Section 5 in [5]). We prove this result by suitably generalizing the argument given in the proof of Theorem 3.10 in [4].
3. the cut admissibility above transfers to the display calculus [1] (in which $\wedge, \vee$ do not have structural counterparts) for the basic normal lattice logic in the signature $(\mathcal{F},\mathcal{G})$ and to its extensions with arbitrary analytic structural rules (cf. Definition [5]). This result follows from the previous one and the fact that every analytic structural rule is equivalent to a set of generalized simple rules (cf. Proposition 60 in [5]).

In this talk, we will also discuss applications of these result such as the finite model property and finite embeddability property for axiomatic extensions of modal expansions of full Lambek calculus.

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[5] For any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$), we let $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) denote the arity of f (resp. g), and the order-type ϵ_f (resp. ϵ_g) on n_f (resp. n_g) indicate whether the i−th coordinate of f (resp. g) is positive ($\epsilon_f(i) = 1$, $\epsilon_g(i) = 1$) or negative ($\epsilon_f(i) = \partial$, $\epsilon_g(i) = \partial$). The order-theoretic motivation for this partition is that the algebraic interpretations of $\mathcal{F}$-connectives (resp. $\mathcal{G}$-connectives), preserve finite joins (resp. meets) in each positive coordinate and reverse finite meets (resp. joins) in each negative coordinate.

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Reconstructing an orthomodular poset from its poset of Boolean subalgebras

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In this contribution we consider the set $\mathcal{B}(P)$ of Boolean subalgebras of an orthomodular poset P. We order $\mathcal{B}(P)$ by inclusion and aim to construct an orthomodular poset isomorphic to P from $\mathcal{B}(P)$ and its order structure.

$\mathcal{B}(P)$ is related to posets of commuting subalgebras of operator algebras. For instance given a C*-algebra A, one can consider the poset $\mathcal{C}(A)$ of commutative C*-subalgebras [1, 7, 12, 13]. Similarly, a von Neumann algebra M has been studied by means of its poset $\mathcal{V}(M)$ of commutative von Neumann subalgebras [4, 6]. The set $\mathrm{Proj}(A)$ of projections in a C*-algebra A forms an orthomodular poset, which is Boolean if A is commutative. This leads to connections between $\mathcal{C}(A)$ and $\mathcal{B}(\mathrm{Proj}(A))$ as exposed in [14, Chapter 6]. $\mathcal{B}(P)$ can also be regarded as a poset of 'commutative subalgebras', since any subset $B \subseteq P$ containing 0 and 1, and that is closed under meets, joins, and the orthocomplementation is a Boolean algebra if and only if each pair p and q of elements in B commute in the following sense: there are pairwise orthogonal $e_1, e_2, e_3 \in B$ such that $p = e_1 \vee e_3$ and $q = e_2 \vee e_3$.

The origins of the study of $\mathcal{B}(P)$ lie in the work of Sachs, who studied the special case of P being a Boolean algebra, proving that $\mathcal{B}(P)$ determines Boolean algebras P up to isomorphism [15]. Only relatively recently Sachs' result was generalized by Harding and Navara in [9] to orthomodular posets as follows. Given two orthomodular posets P and Q, they showed that every order isomorphism $\mathcal{B}(P) \to \mathcal{B}(Q)$ is induced by an orthomodular isomorphism $P \to Q$, which is unique if P does not have blocks (i.e. maximal Boolean subalgebras) of precisely four elements.

As an open problem Harding and Navara stated the direct construction of an orthomodular poset in terms of $\mathcal{B}(P)$ that is isomorphic to P. A partial solution for the case of atomic orthomodular lattices was given by Constantin and Döring in [3]. Our main contribution is a solution to this problem for orthomodular posets P satisfying some mild condition on the center of P (i.e, the set of all $p \in P$ that commute with every element in P):

Theorem 1. Let P be an orthomodular poset whose center has more than four elements. Then we can construct an orthomodular poset in terms of $\mathcal{B}(P)$ that is isomorphic to P.

To sketch a proof of the theorem, we first assume that P is Boolean, and consider the *principal ideal subalgebras* of P, i.e., Boolean subalgebras of P of the form $\downarrow p \cup \uparrow p^\perp$, where $\downarrow p = \{q \in P : q \leq p\}$, and $p^\perp$ is the orthocomplement of p. An order-theoretic characterization of these subalgebras as elements of $\mathcal{B}(P)$ is given in [9, Proposition 3.4]. Since $\downarrow p \cup \uparrow p^\perp = P$ for both $p = 1$ and co-atoms p, principal ideal subalgebras are not sufficient for reconstructing P from $\mathcal{B}(P)$. A solution is to consider the set $\mathcal{P}(P)$ of pairs (D_0, D_1) of principal ideal subalgebras satisfying at least one of the following conditions:

(P0) $D_0 = \{0, 1\}$ and $D_1 = P$;

(P1) $D_0 = P$ and $D_1 = \{0, 1\}$;

(P2) D_0 is an atom of $\mathcal{B}(P)$, and $D_1 = P$;

(P3) $D_0 = P$, and D_1 is an atom of $\mathcal{B}(P)$;

(P4) $D_0 \cap D_1$ is an atom of $\mathcal{B}(P)$, and $D_0 \cap D_1$ is not an principal ideal subalgebra.

Furthermore, we define a binary relation on $\mathcal{P}(P)$, denoted by $\leq$, and defined by $(D_0, D_1) \leq (E_0, E_1)$ if and only if all of the following conditions hold:

(O1) $D_0 \subseteq E_0$;

(O2) $E_1 \subseteq D_1$;

(O3) If (D_0, D_1) satisfies (P2) and (E_0, E_1) satisfies (P3), then $D_0 \neq E_1$.

Moreover, we define a map $\mathcal{P}(P) \to \mathcal{P}(P)$, denoted by $(D_0, D_1) \mapsto (D_0, D_1)^\perp$, defined by $(D_0, D_1) = (D_1, D_0)$. Then one can show that there is a bijection $\Psi_P : P \to \mathcal{P}(P)$ defined by $\Psi_P(p) = (\downarrow p \cup \uparrow p^\perp, \downarrow p^\perp \cup \uparrow p^\perp)$ such that $p \leq q$ if and only if $\Psi_P(p) \leq \Psi_P(q)$ and such that $\Psi_P(p^\perp) = \Psi_P(p)^\perp$. In other words, $\mathcal{P}(P)$ is a Boolean algebra isomorphic to P. If $B \in \mathcal{B}(P)$ contains more than four elements, then we can construct in the same way a Boolean isomorphism $\Psi_B : B \to \mathcal{B}(B)$, but there is also an order embedding $\Psi_{P,B} : \mathcal{P}(B) \to \mathcal{P}(P)$ that preserves the orthocomplementation, satisfies $\Psi_P = \Psi_{P,B} \circ \Psi_B$, and which has image

$$\mathrm{Im}(\Psi_{P,B}) = \{(D_0, D_1) \in \mathcal{P}(P) : D_0 \cap D_1 \subseteq B\}$$

such that $\Psi_{P,B}\big((B \cap D_0, B \cap D_1)\big) = (D_0, D_1)$ for each $(D_0, D_1) \in \mathrm{Im}(\Psi_{P,B})$.

The next step is to consider an orthomodular poset P. Each maximal element M of $\mathcal{B}(P)$ is exactly a block; the down-set $\downarrow M$ is order isomorphic to $\mathcal{B}(M)$, hence we apply our method of reconstructing Boolean algebras to $\downarrow M$, and find a Boolean algebra in terms of $\mathcal{B}(P)$ that is isomorphic to M. Finally, we glue the resulting Boolean algebras together in order to obtain an orthomodular poset in terms of $\mathcal{B}(P)$ that is isomorphic to P. To be more precise: we define

$$\mathcal{P}(P) = \bigcup_{M \in \max \mathcal{B}(P)} \mathcal{P}(M) \Big/ \sim,$$

where $\sim$ is an equivalence relation on $\bigcup_{M \in \max \mathcal{B}(P)} \mathcal{P}(M)$ defined by $(D_0, D_1) \sim (D_0, D_1)$ for $(D_0, D_1) \in \mathcal{P}(M)$, $(E_0, E_1) \in \mathcal{P}(N)$, $M, N \in \max \mathcal{B}(P)$ if and only if there is a $B \in [C(P), M \cap N]$ (the condition on the center assures that B has more than four points) and a $(F_0, F_1) \in \mathcal{P}(B)$ such that

$$\Psi_{M,B}(F_0, F_1) = (D_0, D_1), \quad \Psi_{N,B}(F_0, F_1) = (E_0, E_1).$$

If $[(D_0, D_1)]$ denotes the equivalence class of (D_0, D_1), we define the order on $\mathcal{P}(P)$ by $[(D_0, D_1)] \leq [(E_0, E_1)]$ if and only if there are $(D_0', D_1') \in [(D_0, D_1)]$ and $(E_0', E_1') \in \left[E\right]$ such that $(D_0', D_1') \leq (E_0', E_1')$. The orthocomplementation on $\mathcal{P}(P)$ is defined by

$$[(D_0, D_1)]^\perp = \left[(D_0, D_1)^\perp\right].$$

One now can show that the map $p \mapsto [\Psi_M(p)]$ where M is a block containing p is an orthomodular isomorphism $P \to \mathcal{P}(P)$.

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Subcompletions of representable relation algebras

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Abstract

The variety of representable relation algebras is closed under canonical extensions but not closed under completions. What variety of relation algebras is generated by completions of representable relation algebras? Does it contain all relation algebras? It contains all representable finite relation algebras, and this paper shows that it contains many non-representable finite relation algebras as well. For example, every Monk algebra with six or more colors is a subalgebra of the completion of an atomic symmetric integral representable relation algebra whose finitely-generated subalgebras are finite.

COMPUTING IN DIRECT PRODUCTS OF ALGEBRAS

PETER MAYR

The subpower membership problem (SMP) for a fixed finite algebra $\mathbf{A}$ has as input tuples $a_1, \ldots, a_k, b$ in A^n. The question is whether b belongs to the subalgebra of $\mathbf{A}^n$ generated by $a_1, \ldots, a_k$. The complexity of this problem is important for the effectiveness of various approaches to represent constraints in constraint satisfaction problems (CSP).

We present an overview of complexity results for SMP and related questions for various structures $\mathbf{A}$. Examples of algebras with tractable, NP-complete, PSPACE-complete, or EXPTIME-complete SMP are known. The main question that remains still open is whether every algebra with cube term (or Mal'cev term) has SMP in P. We show that SMP for every such algebra is in NP. Moreover SMP is in P for every algebra that generates a variety with cube term in which the centralizers of monoliths of subdirectly irreducible algebras are supernilpotent.

This is joint work with A. Bulatov, M. Steindl, and Á. Szendrei.

Another proof of Willard's Finite Basis Theorem and Characterizing Congruence Meet Semidistributivity in the Locally Finite Case

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Abstract

I will give several conditions that characterize congruence meet-semidistributivity for locally finite varieties of algebras, based on recent work by Jovanovic, Markovic, McKenzie, and Moore. I will use these conditions to give a new proof of a finite basis theorem published by Baker, McNulty, and Wang in 2004. This finite basis theorem extends Willard's Finite Basis Theorem. This is joint work with Ross Willard.

Comparing approaches to duality for compact Hausdorff spaces

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Stone duality and its natural relatives like Priestley duality provide valuable tools for the algebraist and algebraic logician. They neatly provide a way to think about semantics of a given logic, for example. They also give us constructions of co-limits by transfering limits of dual spaces. But this story leaves the topologist feeling cold. The spaces that arise from the basic natural duality procedure are zero-dimensional by their construction. So we can think about the dual of some peculiar counter-examples in topology such as Cantor space, but not about non-zero-dimensional spaces such as the real closed unit interval or the sphere. We present a category dual compact Hausdorff spaces that arises properly as a generalizatin of Stone duality. We also compare this "natural" approach to another known dual category – that of de Vries.

To find a natural dual to non-zero-dimensional spaces, we identify the source of zero-dimensionality in Stone duality as the reflexivity of the $\leq$ relation on complemented distributive lattices. This suggests a general strategy for obtaining algebraic duals of non-zero dimensional spaces. In particular, we show that by passing to what we call weakening relations (essentially generalized entailment relations) and splitting idempotents in the resulting category, we can remove the obstruction to finding algebraic duals of non-zero dimensional spaces. The result is a category consisting of disributive lattices equipped with binary relations $\vdash$ that behave in all respects except reflexivity like $\leq$ on a complemented distributive lattices. Morphisms are also binary relations that are compatible with $\vdash$ in an obvious way. Morphims compose via relational composition.

We contrast the duality theory that arises this way with the more familiar de Vries duality. For de Vries, the intuition is that a compact Hausdorff space can be recovered from the complete Boolean algebra of its regular opens together with the *rather below* relation on regular opens. So the objects of the de Vries dual category are *complete* Boolean algebras equipped with a binary relation $\prec$ satifying enough conditions to recover a compact regular frame as the ideals that are round with respect to $\prec$. The dual of a continuous function sends a regular open in the codomain to the regularization of the inverse image. So composition of such a morphism is not concrete function composition. In summary, the objects are complete Boolean algebras with binary relation. And although morphisms are functions, they do not compose as such.

Comparing the two approaches, our proposal starts with the classical duality of Stone and generalizes it via well-known category-theoretic techniques (idempotent splitting, mainly). This results in objects and morphisms that are first-order definable. In contrast, De Vries exploits the insight that there are "enough" regular opens in a compact Hausdorff space. This is fundamentally a spatial insight that requires objects to be complete Boolean algebras that are thus not first-order axiomatizable. Likewise, de Vries morphisms are characterized by reference to the completeness of the objects and have a composition defined by taking certain infinite joins.

An approach to the Dedekind completion of pointfree function rings by means of scales *

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In two recent papers ([1, 3]) we presented the construction of the Dedekind completion of the ring $\mathrm{C}(L)$ of continuous real functions on a frame L in three different ways, respectively in terms of

(1) partial real functions on L,

(2) normal semicontinuous real functions on L, and

(3) Hausdorff continuous functions on L.

In this talk we will take another look at the Dedekind completion of the ring $\mathrm{C}(L)$ [2]. The main purpose will be to present an appropriate unifying framework for the various descriptions, which will suggest yet another description in terms of certain type of functions. To that end, we will introduce a notion of generalized scales and we will take advantage of suitable Galois connections and a general result about Galois connections, showing once more the ubiquity of (Galois) adjunctions between partially ordered sets and their conceptual simplicity and extent.

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*Joint work with Javier Gutiérrez García and Jorge Picado

A primer of quasivariety lattices

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Abstract

Let $L_q(K)$ denote the lattice of subquasivarieties of a quasivariety K. We sketch a new proof that $L_q(K)$ is isomorphic to the lattice of H-closed algebraic subsets of an algebraic lattice with a monoid H of continuous operators. These ideas motivate a construction that represents certain finite lattices as $L_q(K)$ for a quasivariety K. New properties of the equational closure operator on $L_q(K)$ are found.

These results are join work with Kira Adaricheva, Jennifer Hyndman and Joy Nishida.

On the undecidability of standardness

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Let $\mathcal{H}$ be a universal Horn class of algebras. The *Boolean core of* $\mathcal{H}$ is the topological quasivariety $\mathcal{H}_{BC}$ generated by the finite members of $\mathcal{H}$ (treated as topological structures with discrete topology). It means that $\mathcal{H}_{BC}$ is the class of topological algebras which are closed subalgebras of products, with nonempty indexing sets, of finite members of $\mathcal{H}$. Notably, $\mathcal{H}_{BC}$ consists of all profinite structures built, as inverse limits, from finite members of $\mathcal{H}$ [4].

In papers [2, 3, 4] the problem of axiomatization of $\mathcal{H}_{BC}$ was undertaken. In general, Boolean cores may behave badly with respect to this issue. Every member of $\mathcal{H}_{BC}$ has a Boolean compatible topology. We call such algebras *Boolean topological algebras*. It may be the case that $\mathcal{H}_{BC}$ is even not definable by a set of first order sentences within the class of all Boolean topological algebras of the type under consideration [4]. However, we may have also a very satisfactory situation when $\mathcal{H}_{BC}$ consists exactly of Boolean topological algebras with algebraic reducts in $\mathcal{H}$. If it is the case, we say that $\mathcal{H}$ is *standard*. In [3] the following problem was formulated.

Problem 1 ([3])**.** Is there an algorithm to decide if a given finite algebra of finite type generates a standard universal Horn class?

The problem is still open. Nevertheless, we propose a solution of its varietal variant. Namely, we prove the following fact.

Theorem 2. *There is no algorithm to decide if a given finite algebra of finite type generates a standard variety.*

It is known that the standardness for a variety $\mathcal{V}$ follows from having *finitely determined syntactic congruences* (FDSC for short) [3]. It means that there is a finite set F of terms $t(x, \bar{y})$ such that for every algebra $\mathbf{A} \in \mathcal{V}$ and every equivalence relation θ on the carrier A of $\mathbf{A}$ the relation $\{(a, a') \in A^2 \mid \forall t \in F, \bar{b} \in A^* \ (t(a, \bar{b}), t(a', \bar{b})) \in \theta\}$ is a congruence of $\mathbf{A}$. This motivated the authors of [3] to formulate also the following related problem.

Problem 3 ([3])**.** Is there an algorithm to decide if a given finite algebra of finite type generates a variety with FDSC?

We give an answer.

Theorem 4. *There is no algorithm to decide if a given finite algebra of finite type generates a variety with FDSC.*

Let us describe briefly how Theorems 2 and 4 were obtained. Recall that a first order positive existential formula $\Gamma(u, v, x, y)$ is called a *congruence formula* provided that $\Gamma(u, v, x, x) \to u = v$ holds in all algebras of the type under consideration. A variety $\mathcal{V}$ has *definable principal subcongruences* (DPSC for short) provided that there is a congruence formula $\Gamma(u, v, x, y)$ such that for every $\mathbf{A} \in \mathcal{V}$ and every pair a, b of distinct elements from A there are distinct elements $c, d \in A$ such that $\mathbf{A} \models \Gamma(c, d, a, b)$ and $\{(e, f) \in A^2 \mid \mathbf{A} \models \Gamma(e, f, c, d)\}$ is a congruence of $\mathbf{A}$ [1]. Recently, based on the work of McKenzie [5], Moore obtained the following result.

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Theorem 5 ([6]). *There is no algorithm to decide if a given finite algebra of finite type generates a variety with DPSC.*

We derived Theorems 2 and 4 from Moore's theorem, some results from [3], and the following our proposition.

Proposition 6. *Let $\mathcal{V}$ be a variety. If $\mathcal{V}$ has DPSC, then $\mathcal{V}$ has FDSC.*

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AN ALGEBRAIC APPROACH TO PROMISE CONSTRAINT SATISFACTION

JAKUB OPRŠAL

The well-known algebraic approach to constraint satisfaction problem (CSP) has brought a lot of results culminating by recent claims of Bulatov and Zhuk of complete characterization of complexity of CSPs with a fixed domain up to log-space reductions. The promise constraint satisfaction problem (PCSP) is a generalization of CSP. The domain of a PCSP is not one, but two relational structures $\mathbb{A}, \mathbb{B}$ in the same language. Usually, it is necessary that there is a homomorphism from $\mathbb{B}$ to $\mathbb{A}$. The promise CSP refer to two closely related problems:

(1) Given a structure in the same language which has a homomorphism to $\mathbb{B}$, find a homomorphism to $\mathbb{A}$.
(2) Given a structure $\mathbb{C}$ in the same language, decide between two cases: there is a homomorphism from $\mathbb{C}$ to $\mathbb{B}$, or there is no homomorphism from $\mathbb{C}$ to $\mathbb{A}$.

An example of such a problem would be approximate (n, k) coloring (the corresponding structures are an n-clique and a k-clique for $k > n$): Given a graph that is n-colorable, find a k-coloring of this graph. This approximate graph coloring is a well known problem in theoretical computer science. These and several other similar problems were recently studied is by Brakensiek, Guruswami, Austrin, and Håstad. They rediscovered a Galois correspondence between pairs of relational structures and minor closed sets of functions that was before described by Pippenger. Nevertheless, while the Galois correspondence is a key ingredient, it lacks the full power of the algebraic approach, namely an analogue to clone homomorphisms. In the talk we will describe the full analogue of the algebraic approach to PCSP, and explain some hardness results of Brakensiek and Guruswami using purely algebraic means. This new approach also explains why the complexity of the standard CSP depends only on linear identities.

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Congruence FD-maximal algebras

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Abstract

We consider the problem of describing the congruence lattices of finite algebras in congruence-distributive varieties. We concentrate on the following special case.

A variety $\mathcal{V}$ is called *congruence FD-maximal*, if for every finite distributive lattice L the following two conditions are equivalent:

(i) L is isomorphic to Con B for some $B \in \mathcal{V}$;

(ii) for every meet-irreducible $x \in L$, the lattice $\uparrow x$ is isomorphic to Con T for some (subdirectly irreducible) $T \in \mathcal{V}$.

(Notice that (i) always implies (ii).)

The concept of congruence FD-maximality can also be considered for individual algebras, in the following sense.

Let A be a finite subdirectly irreducible algebra generating a CD variety. We say that A is *congruence FD-maximal*, if for every finite distributive lattice L the following two conditions are equivalent:

(i) L is isomorphic to Con B for some $B \in P_s H(A)$;

(ii) for every meet-irreducible $x \in L$, the lattice $\uparrow x$ is isomorphic to Con T for some (subdirectly irreducible) $T \in H(A)$.

In other words, A is congruence FD-maximal iff the class of all finite members of Con $P_s H(A)$ is as large as possible by the necessary condition.

The study of congruence FD-maximal algebras is an essential part of the study of congruence FD-maximal varieties. We consider some special types of congruence distributive varieties and present a criterion for them, characterizing the congruence FD-maximality.

Latarres on complete lattices

Wim Ruitenburg, Marquette University

This is a joint project with Mohammad Ardeshir, Sharif University of Technology.

A *latarre* is a lattice with meet $\sqcap$, join $\sqcup$, and an arrow $\rightarrowtriangle$ satisfying the schemas

$x \rightarrowtriangle y = (x \sqcup y) \rightarrowtriangle y.$
$x \rightarrowtriangle y = x \rightarrowtriangle (x \sqcap y).$
$y \trianglelefteq z$ implies $x \rightarrowtriangle y \trianglelefteq x \rightarrowtriangle z.$
$y \trianglelefteq z$ implies $z \rightarrowtriangle x \trianglelefteq y \rightarrowtriangle x.$
$(x \rightarrowtriangle y) \sqcap (y \rightarrowtriangle z) \trianglelefteq x \rightarrowtriangle z.$

where $\trianglelefteq$ is the usual order definable by $x \trianglelefteq y$ exactly when $x \sqcap y = x$. We establish a "decomposition" theorem of a latarre on lattices including frames, into a Heyting latarre and a Löb fixed point latarre. We will include a brief introduction to latarres, with examples.

Local finiteness of modal algebras in terms of Kripke frames partitions

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A logic is locally tabular if it has only finitely many pairwise non-equivalent formulas in each of its finite-variable fragments. Algebraically, modal logics are equational theories of Boolean algebras with operators, thus a modal logic L is locally tabular iff the variety of L-algebras is locally finite. Every locally tabular logic is Kripke complete, i.e., it is the set of formulas valid in a class of relational structures (Kripke frames). Recently, in our joint work with Valentin Shehtman, it was shown that local tabularity of unimodal logics can be characterized in terms of partitions of Kripke frames. In my talk, I will formulate these results, their generalizations for the polymodal case, and discuss some of their corollaries and related open problems.

Transpositional sequences and their multigraphs

Donald Silberger*

June 30, 2017

Abstract

If $\mathbf{s} := \langle s_0, s_1, \dots s_{k-1}\rangle$ is a sequence of length $|\mathbf{s}| = k$ of permutations on the set $n := \{0, 1, \dots n-1\}$, then $\bigcirc\mathbf{s} := s_0 \circ s_1 \circ \cdots \circ s_{k-1} \in \mathrm{Sym}(n)$, and $\mathrm{Seq}(\mathbf{s})$ is its set of *rearrangements*, $\mathbf{r} := \langle s_{\psi(0)}, \dots s_{\psi(k-1)}\rangle$ with $\psi \in \mathrm{Sym}(k)$. Our subject is the set $\mathrm{Prod}(\mathbf{s}) := \{\bigcirc\mathbf{r} : \mathbf{r} \in \mathrm{Seq}(\mathbf{s})\} \subseteq \mathrm{Sym}(n)$. We focus mainly on sequences which are *transpositional*; that is, the terms of $\mathbf{s}$ are transpositions. For $\mathbf{t}$ a transpositional sequence in $\mathrm{Sym}(n)$, there is a natural correspondence between $\mathrm{Seq}(\mathbf{t})$ and its *transpositional multigraph* $\mathcal{T}(\mathbf{t}) := \langle n; E(\mathbf{t})\rangle$ on the vertex set n, where the k simple edges $(a\, b) \in E(\mathbf{t})$ of $\mathcal{T}(\mathbf{t})$ are the k terms of $\mathbf{t}$.

We study two special sorts of transpositional sequences: We call $\mathbf{t}$ and $\mathcal{T}(\mathbf{t})$ *permutationally complete*, abbreviated perm-complete, iff $\mathrm{Prod}(\mathbf{t}) \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\backslash\mathrm{Alt}(n)\}$, where $\mathrm{Alt}(n)$ is the group of all even permutations of the set n. I.e., if $\mathbf{t}$ is perm-complete then $\mathrm{Prod}(\mathbf{t})$ is as large as possible.

We call $\mathbf{t}$ and $\mathcal{T}(\mathbf{t})$ *conjugacy invariant*, aka CI, iff the elements in $\mathrm{Prod}(\mathbf{t})$ are mutually conjugate. $\mathrm{Prod}(\mathbf{t})$ is small if $\mathbf{t}$ is CI. We specify the CI transpositional sequences in $\mathrm{Sym}(n)$, and initiate the study of those sequences $\mathbf{s}$ in $\mathrm{Sym}(n)$, each of whose terms has exactly one nontrivial cyclic component.

The talk reports on a paper-in-progress which combines results of the presenter together with those of his students, Raymond R. Fletcher [6] and Arthur D. Tuminaro [9], and of Fletcher's student, Alissa R. Ellis [5], which extend the work of J. Dénes [3] and of M. Eden with M. P. Schützenberger [4].

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Compactness in the universe

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Abstract

A structure is an example of compactness if whenever every substructure of smaller cardinality satisfies a certain property, then the whole structure satisfies it, too.

Key compactness type principles are the tree property and failure of square. An old project in set theory is to consistently get the tree property at every regular cardinal greater than ω_1. Doing so requires large cardinals and forcing. We will survey some classical and recent results on obtaining instances of compactness like the tree property. We will also go over ZFC constraints, highlighting the challenges of obtaining these combinatorial principles, especially at successors of singular cardinals.

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There are $2^{\aleph_0}$ pre-maximal extension of the relevant logic **E**

Entailment logic **E** was presented by A.R. Anderson and N.D. Belnap in 1975 [1]. However, not much is known about the structure of extension of the logic **E**; the logic **R** and **RM** and their extensions were a subject of deep investigations [3],[5],[6],[7],[8]. What has been shown about **E** is the lack of algebraizability [2]. We also know that the logic **E** is not structurally complete [4].

In this talk we try characterize the structure of the lattice of the extension of the logic **E** (without constants). We devote our special attention to the upper part of the lattice of the extension of **E**. It turns out that there are $2^{\aleph_0}$ coatoms in the interval $[\mathbf{E}, \mathbf{CL}]$, where **CL** denotes the classical logic, while the interval $[\mathbf{R}, \mathbf{CL}]$ contains 3 coatoms, and the interval $[\mathbf{RM}, \mathbf{CL}]$ contains only one coatom.

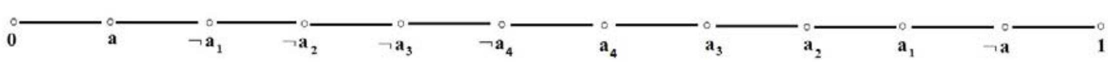
The extension of the logic E is called *pre-maximal* if and only if it is coatom in the interval $[\mathbf{E}, \mathbf{CL}]$ (of course the maximal extension of the logic **E** is **CL**).

Theorem 1. *There are $2^{\aleph_0}$ pre-maximal extension of the relevant logic* **E**.

We present an infinite binary tree of simple, finite **E**-algebras. The nodes of this tree are algebras based on finite chains. Each branch of the tree represents an infinite denumerable **E**-algebra.

Let us add some details. The structure of the binary tree in question can be described by induction.

Step 0 (level 0). The algebra from the level 0 (algebra $\mathbf{A^0}$) is based on a 12-element chain



and the operation $\to$ is defined in the table below:

$\to$	a	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$	a_4	a_3	a_2	a_1	$\neg a$
a	a	a	a	a	a	a_4	a_3	a_2	a_1	a_1
$\neg a_1$	0	a	a	a	a	a_4	a_3	a_2	a_2	a_1
$\neg a_2$	0	0	a	a	a	a_4	a_3	a_3	a_2	a_2
$\neg a_3$	0	0	0	a	a	a_4	a_4	a_3	a_3	a_3
$\neg a_4$	0	0	0	0	a	a_4	a_4	a_4	a_4	a_4
a_4	0	0	0	0	0	a	a	a	a	a
a_3	0	0	0	0	0	0	a	a	a	a
a_2	0	0	0	0	0	0	0	a	a	a
a_1	0	0	0	0	0	0	0	0	a	a
$\neg a$	0	0	0	0	0	0	0	0	0	a

The set of designated values of this algebra is $[a) = \{x : a \leq x\}$ (it is true for all the algebras we consider), thus $\mathbf{A^0}$ is a simple algebra.

Step 1 (level 1). We construct two new algebras (i.e. $\mathbf{A^{00}}$ and $\mathbf{A^{01}}$) based on $\mathbf{A^0}$. We add new elements $a_5, \neg a_5$ to the old ones; now we have the chain $0 < \ldots < \neg a_3 < \neg a_4 < \neg a_5 < a_5 < a_4 < a_3 < \ldots < 1$; the new chain has 14 elements. Next, we define the operation $\rightarrow$ in $\mathbf{A^{00}}$ and $\mathbf{A^{01}}$ in the following way. The values of $\rightarrow$ for elements $0, \ldots, \neg a_3$ and their negations and for $\neg a_4$ remain the same as in $\mathbf{A^0}$. In $\mathbf{A^{00}}$ we set $a_5 = \neg a_1 \rightarrow a_4$ and in $A^0 1$ we set $a_5 = \neg a_4 \rightarrow a_4$; in consequence the values for $x \rightarrow a_4$ and $\neg a_4 \rightarrow y$ must be changed in both algebras.

$\mathbf{Step\ n+1}$ (level $n+1$). Let us consider the algebras from the level n; we denote them by $\mathbf{A^n}$ where n stands for the 0-1 sequence of the length n with 0 as the first element. The $\mathbf{A^n}$-algebras are based on the $(12+2n)$-element chain. Each algebra $\mathbf{A^n}$ determines two algebras: ($\mathbf{A^{n0}}$ and $\mathbf{A^{n1}}$) from the level $n+1$. Algebras from the level $n+1$ are based on the $(12+2(n+1))$-element chain, in which $0 < \ldots < \neg a_n < \neg a_{n+1} < a_{n+1} < a_n < \ldots < 1$. As in the step 1, the definition of $\rightarrow$ in $\mathbf{A^{n+1}}$-algebras is based on the definition of $\rightarrow$ in $\mathbf{A^n}$. Let $\mathbf{A^n}$ be fixed. Then in $\mathbf{A^{n0}}$ we set $a_{n+1} = \neg a_1 \rightarrow a_n$, and in $\mathbf{A^{n1}}$ we set $a_{n+1} = \neg a_n \rightarrow a_n$. The values for $0, \ldots, \neg a_{n-1}$ and its negations, and for $\neg a_n$ remain the same as in the algebra $\mathbf{A^{n-1}}$, which precedes the algebra $\mathbf{A^n}$, and the values of $x \rightarrow a_n$ and $\neg a_n \rightarrow y$ must be changed.

Now, if we consider a branch of this tree, we get an infinite denumerable **E**-algebra; the operation $\rightarrow$ can be reconstructed from the $\mathbf{A^n}$-algebras in this branch.

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CRYPTOGRAPHIC APPLICATIONS OF VERY LARGE CARDINALS

JOSEPH VAN NAME

A self-distributive algebra is an algebra $(X,*)$ that satisfies the identity $x*(y*z)=(x*y)*(x*z)$.

Suppose $(X,*)$ is self-distributive. An element $x\in X$ is said to be a left-identity if $x*y=y$ for all $y\in X$. We say that a subset $L\subseteq X$ is a left-ideal if $y\in L$ implies $x*y\in L$. Let $\mathrm{Li}(X)$ denote the set of all left-identities in X.
We say that $(X,*)$ is Laver-like if

(1) $\mathrm{Li}(X)$ is a left-ideal in X, and
(2) whenever $x_n\in X$ for $n\in\omega$, there is some $N\in\omega$ with $x_0*...*x_N\in\mathrm{Li}(X)$ (Parentheses are grouped on the left. i.e. $x*y*z=(x*y)*z$).

If $(X,*)$ is a Laver-like algebra, then define the Fibonacci terms t_n for $n\geq 1$ by letting $t_1(x,y)=y, t_2(x,y)=x$, and $t_{n+2}(x,y)=t_{n+1}(x,y)*t_n(x,y)$. Then for all x,y there is some n where $t_n(x,y)\in\mathrm{Li}(X)$. Define an associative operation $\circ$ on $X\setminus\mathrm{Li}(X)$ by letting $x\circ y=t_{n+1}(x,y)$ where n is chosen such that $t_n(x,y)\in\mathrm{Li}(X)$.

The existence of a non-trivial elementary embedding $j:V_\lambda\rightarrow V_\lambda$ is among the strongest of all large cardinal axioms.

Let $\mathcal{E}_\lambda$ be the set of all elementary embeddings $j:V_\lambda\rightarrow V_\lambda$. Then define an operation $*$ on $\mathcal{E}_\lambda$ by letting $j*k=\bigcup_{\alpha<\lambda}j(k|_{V_\alpha})$. $(\mathcal{E}_\lambda,*)$ is self-distributive.

If γ is a limit ordinal with $\gamma<\lambda$, then define a congruence $\equiv^\gamma$ on $(\mathcal{E}_\lambda,*)$ by letting $j\equiv^\gamma k$ iff $j(x)\cap V_\gamma=k(x)\cap V_\gamma$ for all $x\in V_\gamma$. Then $(\mathcal{E}_\lambda/\equiv^\gamma,*)$ is a Laver-like algebra.

If n is a natural number, then there is a unique algebra $A_n=(\{1,...,2^n\},*)$ such that

(1) $x*(y*z)=(x*y)*(x*z)$, and
(2) $x*1=x+1 \mod 2^n$ for all x,y,z

which we shall call a classical Laver table.

Every Laver-like algebra generated by a single element is isomorphic to some A_n.

Suppose that t is an $n+1$-ary operation on a set X. Then t is said to be self-distributive if it satisfies the identity

$$t(x_1,...,x_n,t(y_1,...,y_n,y))$$
$$=t(t(x_1,...,x_n,y_1),...,t(x_1,...,x_n,y_n),t(x_1,...,x_n,y)).$$

If (X,t) is an $n+1$-ary self-distributive algebra, then define the hull $\Gamma(X,t)=(X^n,*)$ where $*$ is the binary operation defined by $(x_1,...,x_n)*(y_1,...,y_n)=(t(x_1,...,x_n,y_1),...,t(x_1,...,x_n,y_n))$. Then $\Gamma(X,t)$ is a self-distributive algebra. We say that an $n+1$-ary self-distributive algebra (X,t) is Laver-like if its hull $\Gamma(X,t)$ is Laver-like.

$n+1$-ary Laver-like algebras can easily be produced from the algebras of the form $\mathcal{E}_\lambda/\equiv^\gamma$.

In the following key exchange, Alice and Bob want to share a common secret by communicating over a public channel. A semigroup $(X, \circ)$ and an element $x \in X$ are known to the public.

(1) Alice selects some $a \in X$ and then sends $r = a \circ x$ to Bob.
(2) Bob selects some $b \in X$ and the sends $s = x \circ b$ to Alice.
(3) Let $K = a \circ x \circ b$.
(4) Alice computes K using the fact that $K = a \circ s$ and Alice knows a, s.
(5) Bob computes K using the fact that $K = r \circ b$ and Bob knows r, b.

An eavesdropping party will only know x, r, s. No eavesdropping party should be able to compute K with this information. Therefore K is a shared secret between Alice and Bob established over a public channel.

Suppose that $(X, t^{\bullet})$ is an $n+1$-ary Laver-like algebra. Then let $\Diamond(X, t^{\bullet})$ be the algebra whose underlying set consists of all functions $\mathfrak{l} : \{1, ..., n\}^* \rightarrow X \cup \{\#\}$ that satisfies the following

(1) $\mathfrak{l}(\varepsilon) \in X$
(2) $\mathfrak{l}(\mathbf{x}) \in X$ for only finitely many $\mathbf{x} \in \{1, ..., n\}^*$
(3) If $\mathfrak{l}(\mathbf{x}) = \#$ then $\mathfrak{l}(i\mathbf{x}) = \#$
(4) If $\mathfrak{l}(\mathbf{x}) \in X$ then either $\mathfrak{l}(i\mathbf{x}) = \#$ for $1 \leq i \leq n$ or $\mathfrak{l}(i\mathbf{x}) \in X$ for $1 \leq i \leq n$.
(5) If $\mathfrak{l}(1\mathbf{x}) \in X$, then there is some $x \in X$ where $t^{\bullet}(\mathfrak{l}(1\mathbf{x}), ..., \mathfrak{l}(n\mathbf{x}), x) = \mathfrak{l}(\mathbf{x})$
(6) If $\mathfrak{l}(1\mathbf{x}) \in X$, then $(\mathfrak{l}(1\mathbf{x}), ..., \mathfrak{l}(n\mathbf{x})) \in \mathbf{Li}(\Gamma(X, t^{\bullet}))$

The set $\Diamond(X, t^{\bullet})$ can be endowed with a unique operation $t^{\sharp}$ such that

(1) if $(\mathfrak{l}_1(\varepsilon), ..., \mathfrak{l}_n(\varepsilon)) \in \mathrm{Li}(\Gamma(X, t^{\bullet}))$, then $t^{\sharp}(\mathfrak{l}_1, ..., \mathfrak{l}_n, \mathfrak{l}) = \mathfrak{l}$,
(2) $t^{\sharp}(\mathfrak{l}_1, ..., \mathfrak{l}_n, \mathfrak{l}) = t_{\mathfrak{l}(\varepsilon)}(\mathfrak{l}_1, ..., \mathfrak{l}_n)$ whenever $\mathfrak{l}(1) = \#$, and
(3) $t^{\sharp}(\mathfrak{l}_1, ..., \mathfrak{l}_n, t_x(\mathfrak{u}_1, ..., \mathfrak{u}_n)) = t^{\sharp}(t^{\sharp}(\mathfrak{l}_1, ..., \mathfrak{l}_n, \mathfrak{u}_1), ..., t^{\sharp}(\mathfrak{l}_1, ..., \mathfrak{l}_n, \mathfrak{u}_n), t_x(\mathfrak{l}_1, ..., \mathfrak{l}_n)$.

Then $(\Diamond(X, t^{\bullet}), t^{\sharp})$ is an $n+1$-ary Laver-like algebra called a functional endomorphic Laver table.

Furthermore, if there are efficient algorithms for computing $\mathfrak{l}_1, ..., \mathfrak{l}_n, \mathfrak{l}$, then one can also compute $t^{\sharp}(\mathfrak{l}_1, ..., \mathfrak{l}_n, \mathfrak{l})(\mathbf{x})$ efficiently.

If $(X, t^{\bullet})$ is an $n+1$-ary Laver-like algebra, then $(\Gamma(\Diamond(X, t^{\bullet})) \backslash \mathrm{Li}((\Gamma(\Diamond(X, t^{\bullet}))), \circ)$ may be used as a platform for the Ko-Lee key exchange.

In year 1999, Anshel, Anshel, and Goldfeld have constructed a key exchange which could use any non-abelian group as a platform. In 2013, Kalka and Teicher have constructed a self-distributive version of the Anshel-Anshel-Goldfeld key exchange. This key exchange by Kalka and Teicher extends to n-ary self-distributive algebras as well. The functional endomorphic Laver tables may be used as a platform for this key exchange.

In 2006, Dehornoy has shown that self-distributive algebras may be used as platforms for authentication schemes. In particular, the functional endomorphic Laver tables may be used as platforms for such authentication schemes.

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The finite basis problem, Jónsson's speculation, and weird algebras

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Abstract

The finite basis problem in universal algebra is basically the following question: for which finite algebras A in a finite signature is the set of equational identities true in A finitely axiomatizable? Are there structural properties of A which guarantee a positive or negative answer?

This problem has a long history, though interest in the problem has faded to the background in recent years. In this tutorial I will aim to (re-)introduce the problem through examples, describe some important partial solutions, and state some open problems. I will pay particular attention to a structural question of Bjarni Jónsson, and to the question of existence of "bizarrely nonfinitely based" algebras.

Stonian p-Ortholattices for Pointless Topology

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Abstract. In this presentation we will investigate Stonian p-ortholattices as an algebraic approach to pointless topology. We will focus on their relationship to Boolean Contact algebras and their representation theory.

Stonian p-ortholattice were introduced [5] as the algebraic counterpart to the the mereotopology RT_0 of Asher and Vieu [1]. A Stonian p-ortholattice is a lattice with three complement operations, a pseudo-complement, an orthocomplement, and the dual of the pseudo-complement, also called quasi-complement, satisfying a Stone-like equation. Mereotopology is a branch of pointless topology and a composition of the topological notion of connectedness (or being in contact) with the mereological notion of parthood. In this approach regions are considered basic entities rather than specific sets of points. In the case of RT_0, or equivalently Stonian p-ortholattices, the intended model, i.e., the set of regions, is the set of regular sets of a topology. A regular set is a set so that its closure is regular closed and its interior is regular open, i.e., avoiding "isolated points" and "cracks".

Another lattice based approach to mereotopology is given by Boolean Contact Algebras (BCAs), i.e., Boolean algebras with an additional contact relation C satisfying certain axioms. These lattices correspond to the Region Connection Calculus [2, 3, 6]. The intended model for these structures is the set of regular closed sets of a topology. In fact, in [4] a representation theorem by the regular closed sets for BCAs was shown.

In this talk we present the basic theory of Stonian p-ortholattices. Furthermore, we exhibit the relationship between Boolean Contact Algebras (BCAs) and Stonian p-ortholattices by using the skeleton of Stonian p-ortholattices as bridging structure. We show that the skeleton $S(L)$ of an arbitrary Stonian p-ortholattice L is a BCA when defining the contact relation of the BCA in terms of the lattice L. On the reverse, we prove that every BCA can be embedded in a Stonian p-ortholattice. Last but not least, we will address the representation problem by regular sets for this class of lattices. First, introduce five properties (RP_1), (RP_2), (M), (S), as well as a localized version of distributivity (D), that are always satisfied by lattices constructed from the regular sets of a topological

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space. All five properties can be expressed as quasiidentities, thus preserving the equational character if we extend the theory of Stonian p-ortholattices by those properties. Then we provide an example of a Stonian p-ortholattice that is not representable contradicting the completeness claim of RT_0 in [1]. An actual representation theorem is still outstanding.

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Characterizing Supernilpotent Algebras

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Abstract

An algebra is supernilpotent if it is abelian in the sense of the higher commutator defined by Andrei Bulatov in his work on clones with a Mal'cev term. In the case of Mal'cev algebras, there is a characterization of supernilpotent algebras which generalizes the affine structure of abelian algebras (in the sense of the binary term-condition commutator) and the expanded group structure of 3-supernilpotent algebras. As in these cases, there is an "ideal" class of algebras which are the abelian models in the higher commutators.

For the theory of the higher commutator in arbitrary varieties, I'll discuss the possibility that (1) neutrality of the higher commutators, and (2) supernilpotent algebras interpreting a Mal'cev term both define Mal'cev conditions.

Properties of crucial CSP instances.

Dmitriy Zhuk

Let Γ, called *a constraint language*, be a set of relations (or predicates) on a finite set A. Constraint Satisfaction Problem CSP(Γ) can be defined as the following decision problem: given a conjunction of predicates, i.e. a formula

$$\rho_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge \rho_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $\rho_1, \ldots, \rho_s \in \Gamma$; decide whether the formula is satisfiable.

In 1998 it was conjectured that CSP(Γ) is either in P, or NP-complete [2]. Later it was conjectured that CSP(Γ) can be solved in polynomial time if Γ is preserved by a weak near-unanimity operation. Recently, several proofs of this conjecture appeared [4, 1, 6]. In the talk we consider implications of one of the proof.

As we know from [3] the complexity of CSP(Γ) depends only on the relational clone generated by Γ. Thus we assume that Γ is a relational clone preserved by a weak near-unanimity operation.

A CSP instance is called *crucial* if it has no solutions but the replacement of any constraint by all weaker constraints from Γ gives an instance with a solution. It turned out that with few more assumptions (cycle-consistency and irreducibility) every relation of a crucial instance has parallelogram property. The description of critical relations from [5] implies that every relation of a crucial instance can be represented as a union of blocks, where each block is defined by a linear equation. It can be shown that these blocks and linear equations of some constraints with a common variable are related to each other.

Below we give necessary definitions to formulate mathematical statements explaining this idea.

1 Definitions

Suppose Θ is a CSP instance with the set of variables $\mathbf{X} = \{x_1, \ldots, x_n\}$, set of respective domains $\mathbf{D} = \{D_1, \ldots, D_n\}$, set of constraints $\mathbf{C} = \{C_1, \ldots, C_m\}$. Θ is called *subdirect* if its solution set is a subdirect relation in $D_1 \times \cdots \times D_n$. We say that an instance Θ is *fragmented* if the set of variables $\mathbf{X}$ can be divided into 2 nonempty sets $\mathbf{X_1}$ and $\mathbf{X_2}$ such that the constraint scope of any constraint either has variables only from $\mathbf{X_1}$, or only from $\mathbf{X_2}$. A CSP instance is called *cycle-consistent* if for every i and $a \in D_i$, any path starting and ending with x_i in Θ connects a and a. A CSP instance Θ is called *linked* if for every i and $a, b \in D_i$ there exists a path in Θ that connects a and b. A CSP instance Θ is called *irreducible* if for any $\mathbf{C}' \subseteq \mathbf{C}$ and any set of variables $\mathbf{X}' \subseteq \mathbf{X}$ the projection of $\mathbf{C}'$ onto $\mathbf{X}'$ is fragmented, linked, or subdirect.

We say that a relation *has parallelogram property* if any permutation of variables in ρ satisfies the following implication

$$\forall \alpha_1, \beta_1, \alpha_2, \beta_2 \colon (\alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2 \in \rho \Rightarrow \alpha_1\alpha_2 \in \rho).$$

We say that the i-th variable of a relation ρ is *compatible with the congruence* σ if $(a_1, \ldots, a_n) \in \rho$ and $(a_i, b_i) \in \sigma$ implies $(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) \in \rho$.

Suppose σ_1 and σ_2 are congruences on D_1 and D_2, correspondingly. A relation $\rho \in D_1^2 \times D_2^2$ is called *a link* from σ_1 to σ_2 if the first two variables of ρ are compatible with σ_1, the last two variables of ρ are compatible with σ_2, $\mathrm{pr}_{1,2}(\rho) \supsetneq \sigma_1$, $\mathrm{pr}_{3,4}(\rho) \supsetneq \sigma_2$, and $(a_1, a_2, a_3, a_4) \in \rho$ implies $(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2$.

Suppose σ_1, σ_2, σ_3 are congruences such that we have a link ρ_1 from σ_1 to σ_2 and a link ρ_2 from σ_2 to σ_3. Then we can define a link from σ_1 to σ_3 by $\exists y_1 \exists y_2 \rho_1(x_1, x_2, y_1, y_2) \wedge \rho_2(y_1, y_2, z_1, z_2)$.

A link $\rho \subseteq D^4$ is called *reflexive* if $(a, a, a, a) \in \rho$ for every $a \in D$.

We say that two congruences σ_1 and σ_2 on a set D are *adjacent* there exists a reflexive link from σ_1 to σ_2.

Example 1. Since we can always put $\rho(x_1, x_2, x_3, x_4) = \sigma(x_1, x_3) \wedge \sigma(x_2, x_4)$, any congruence σ is adjacent with itself.

Example 2. Trivial congruence and modulo 2 congruence on $\mathbb{Z}_4$ are adjacent because we have the following subalgebra $\{(a_1, a_2, b_1, b_2) \mid a_1 - a_2 = 2b_1 - 2b_2\}$.

For a relation ρ by $\mathrm{Con}(\rho, i)$ we denote the binary relation $\sigma(y, y')$ defined by

$$\exists x_1 \ldots \exists x_{i-1} \exists x_{i+1} \ldots \exists x_n \; \rho(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \wedge \rho(x_1, \ldots, x_{i-1}, y', x_{i+1}, \ldots, x_n).$$

For a constraint $C = \rho(x_1, \ldots, x_n)$, by $\mathrm{Con}(C, x_i)$ we denote $\mathrm{Con}(\rho, i)$. It is easy to see that for a relation ρ with parallelogram property $\mathrm{Con}(\rho, i)$ is always a congruence.

We say that two constraints C_1 and C_2 are *adjacent* in a common variable x if $\mathrm{Con}(C_1, x)$ and $\mathrm{Con}(C_1, x)$ are adjacent. An instance is called *connected* if for every two constraints there exists a path that connects them. It can be shown that every two constraints with common variable in a connected instance are adjacent.

By $\mathrm{Var}(\Omega)$ we denote the set of all variables of the instance Ω. We say that an instance Ω' is *an expansion* of an instance Ω if there exists a mapping $S : \mathrm{Var}(\Omega') \to \mathrm{Var}(\Omega)$ such that for every constraint $\rho(x_1, \ldots, x_n)$ of Ω' the constraint $\rho(S(x_1), \ldots, S(x_n))$ is weaker than some constraint of Ω.

2 Theorems

Theorem 1. *Suppose Θ is a crucial cycle-consistent irreducible CSP instance in a constraint language Γ. Then every constraint relation has parallelogram property.*

Theorem 2. *Suppose Θ is crucial cycle-consistent irreducible CSP instance. Then there exists a crucial expansion of Θ containing a linked connected component that is not subdirect.*

I am questioning whether this theorem can be strengthened in the following form.

Conjecture 1. *Suppose Θ is crucial cycle-consistent irreducible CSP instance. Then Θ is connected.*

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