Around canonical heights in arithmetic dynamics

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Plan of the talk

(2005 Email? 2008 AIM)

1. Canonical heights for polarized dynamical systems
2. Canonical heights on affine space
3. Canonical heights for surface automorphisms
4. Arithmetic degrees

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1. Canonical heights for polarized dynamical systems
2. Canonical heights on affine space
3. Canonical heights for surface automorphisms
4. Arithmetic degrees

The emphasis is on canonical heights other than Néron-Tate heights or those on $\mathbb{P}^n$ for morphisms.

Part 1 Canonical heights for polarized dynamical systems

$\boldsymbol{X}$: a projective variety defined over $\bar{\mathbb{Q}}$ (for simplicity)

$\boldsymbol{f : X \to X}$: a morphism

$\boldsymbol{D} \in \mathrm{Div}(X)_{\mathbb{R}} := \mathrm{Div}(X) \otimes \mathbb{R}$: a Cartier $\mathbb{R}$-divisor on X

Assume that $\boldsymbol{f^*D \sim d\,D}$ for some $d > 1$.

If D is ample, then the triple $\boldsymbol{(X, f, D)}$ is called a **polarized dynamical system**.

Example (polarized dynamical systems)

- X: Abelian variety, D ample with $[-1]^*D \sim D$,
  $f = [2]$: twice multiplication map ($\Longrightarrow$ **Néron-Tate height**)
- $X = \mathbb{P}^N$, f: a morphism of degree > 1, D: a hyperplane
  ($\Longrightarrow$ **canonical height** $\hat{h}_f : \mathbb{P}^N(\bar{\mathbb{Q}}) \to \mathbb{R}$)

Theorem (Call–Silverman 1993)

(D: not necessarily ample.) There exists a unique height function $\hat{\boldsymbol{h}}_{\boldsymbol{D}}$ *associated to D,*

$$\hat{\boldsymbol{h}}_{\boldsymbol{D}} : \boldsymbol{X}(\bar{\mathbb{Q}}) \to \mathbb{R},$$

satisfying $\hat{\boldsymbol{h}}_{\boldsymbol{D}} \circ \boldsymbol{f} = \boldsymbol{d}\,\hat{\boldsymbol{h}}_{\boldsymbol{D}}$.

Properties of Call–Silverman canonical heights

1. Assume that D is ample. Then $\hat{h}_D$ is non-negative and

   $$\hat{\boldsymbol{h}}_{\boldsymbol{D}}(\boldsymbol{x}) = \mathbf{0} \text{ if and only if } x \in X(\bar{\mathbb{Q}}) \text{ is } \textbf{preperiodic}.$$

   In particular, the set of preperiodic points $\mathbf{PrePer}(\boldsymbol{f}, \bar{\mathbb{Q}})$ is a set of bounded height.

2. Assume that D is ample. Take $x \in X(\bar{\mathbb{Q}})$ that is not preperiodic. Then

$$\#\{y \in O_f^+(x) \mid h_H(y) \le T\} \sim \frac{\log T}{\log d} \quad \text{as } T \to \infty,$$

where $O_f^+(x)$ is the forward orbit of x under f, and H is any ample divisor.

3. **Decomposition into the sum of local canonical heights**

Take a number field K over which f is defined.

For a finite extension L/K and $x \in X(L) \setminus |D|$, one has

$$\hat{h}_D(x) = \sum_{v \in M_L} \frac{[L_v : K_v]}{[L : K]} \hat{\lambda}_{D,v}(x).$$

❹ Variation of the canonical height

$\pi : \mathcal{V} \to C$ a family over a smooth projective curve C

C° a Zariski open subset of C, and set $\mathcal{V}^\circ := \pi^{-1}(C^\circ)$

$f : \mathcal{V}^\circ \to \mathcal{V}^\circ$ over C° and $\mathcal{D} \in \mathrm{Div}(\mathcal{V}^\circ)_{\mathbb{R}}$ as before

h_C: a Weil height on C corresponding to a divisor of degree 1

$P : C \to \mathcal{V}$ a section

$$\lim_{h_T(t)\to\infty} \frac{\hat{h}_{\mathcal{D}_t}(P_t)}{h_T(t)} = \hat{h}_{\mathcal{D}}(P).$$

Followed by stronger results by Ingram

More properties of the canonical heights ...

From talks of this conference: Equidistribution (Baker–Rumely, Chambert-Loir, Favre–Rivera-Letlier, Yuan ...), Masser–Zannier unlikely intersection (Baker–DeMarco, Ghioca–Tucker–Hsia, DeMarco–Wang–Ye, Ghioca–Krieger–Nguyen–Ye ...) ...

Part 2 Canonical heights on affine space

Hénon map

$\mathbb{A}^2$: affine plane (over $\bar{\mathbb{Q}}$)

$f : \mathbb{A}^2 \to \mathbb{A}^2$: a **Hénon map**, i.e., an automorphism of the form

$$f(x, y) = (y + P(x), x)$$

for some polynomial $P(x) \in \bar{\mathbb{Q}}[x]$ with $d := \deg(P) \geq 2$.

Then f extends to a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$,

$$(x : y : z) \mapsto (yz^{d-1} + z^d P(x/z) : xz^{d-1} : z^d).$$

Since f has the **indeterminacy set** $I_f = \{(1 : 0 : 0)\}$,
f **is not a morphism.**

The Hénon map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is not a polarized dynamical system, but Silverman proved the following theorem.

Theorem (Silverman 1994)

Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a Hénon map of degree 2 over $\bar{\mathbb{Q}}$. Then

1. *the set of periodic points* $\mathbf{Per}(f, \bar{\mathbb{Q}})$ *is a set of bounded height.*
2. *Take $x \in \mathbb{A}^2(\bar{\mathbb{Q}})$ that is not periodic. Then*

$$\#\{y \in O_f(x) \mid h_{\mathrm{Weil}}(y) \le T\} \sim 2\,\frac{\log T}{\log 2} \qquad \text{as } T \to \infty,$$

   where $O_f(x) = \{f^n(x) \mid n \in \mathbb{Z}\}$ is the f-orbit, and h_{Weil} is the standard Weil function.

Remark The proof uses blow-ups along the indeterminacy sets I_f and $I_{f^{-1}}$.

Regular polynomial automorphism

$\mathbb{A}^N$: affine N-space (over $\bar{\mathbb{Q}}$)

$f : \mathbb{A}^N \to \mathbb{A}^N$: a polynomial automorphism

Then f extends to a birational map $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$.

Following Sibony, f is called a **regular polynomial automorphism**

if
$$I_f \cap I_{f^{-1}} = \emptyset,$$

where I_f and $I_{f^{-1}}$ is the indeterminacy sets of f and f^{-1}.

Example

{Hénon maps} $\subset$ {regular polynomial automorphisms}

Indeed, for a Hénon map, $I_f = \{(1:0:0)\}$ and $I_{f^{-1}} = \{(0:1:0)\}$.

Silverman's results are generalized by Denis and Marcello.

Theorem (Denis, Marcello)

Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism of degree $d \geq 2$. Then

1. *the set of periodic points* $\mathbf{Per}(f, \bar{\mathbb{Q}})$ *is a set of bounded height.*
2. *Take $x \in \mathbb{A}^N(\bar{\mathbb{Q}})$ that is not periodic. Then*

$$\#\{y \in O_f(x) \mid h_{\mathrm{Weil}}(y) \leq T\} \sim 2\,\frac{\log T}{\log d} \quad \text{as } T \to \infty,$$

where $O_f(x) = \{f^n(x) \mid n \in \mathbb{Z}\}$ is the f-orbit, and h_{Weil} is the standard Weil function.

Canonical heights have not appeared so far, but they exist.

For a polynomial automorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ on the affine plane, the **(first) dynamical degree** (explained more later) is defined by

$$\boldsymbol{\delta} := \boldsymbol{\delta}_f := \lim_{n\to\infty} (\deg f^n)^{1/n},$$

which in this case is an integer. (For a Hénon map, $\delta = \deg f$.)

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which in this case is an integer. (For a Hénon map, $\delta = \deg f$.)

Theorem (K. ($\dim N = 2$))

Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial automorphism with $\delta > 1$.
Then the limits

$$\hat{h}^+(x) := \lim_{n\to\infty} \frac{1}{\delta^n} h_{\mathrm{Weil}}(f^n(x)), \quad \hat{h}^-(x) := \lim_{n\to\infty} \frac{1}{\delta^n} h_{\mathrm{Weil}}(f^{-n}(x))$$

exist for all $x \in \mathbb{A}^2(\bar{\mathbb{Q}})$. We set $\boldsymbol{\hat{h} := \hat{h}^+ + \hat{h}^-} : \mathbb{A}^2(\bar{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$.
Then $\hat{h}$ satisfy $\widehat{h} \gg\ll h_{\mathrm{Weil}}$
and $\boldsymbol{\hat{h} \circ f + \hat{h} \circ f^{-1} = \left(\delta + \frac{1}{\delta}\right) \hat{h}}$. Further (... continued)

Properties of canonical heights

1. $$\boldsymbol{\hat{h}(x) = 0} \text{ if and only if } x \in \mathbb{A}^2(\bar{\mathbb{Q}}) \text{ is } \textbf{periodic}.$$

   In particular, the set of periodic points $\mathbf{Per}(f, \bar{\mathbb{Q}})$ is a set of bounded height (Silverman).

2. Take $x \in X(\bar{\mathbb{Q}})$ that is not preperiodic. Then, as $T \to \infty$,

   $$\#\{y \in O_f(x) \mid h_{\mathrm{Weil}}(y) \leq T\} = 2\frac{\log T}{\log \delta} - \widehat{h}(O_f(x)) + O(1) \quad ,$$

   where $\widehat{h}(O_f(x))$ is a quantity defined by the orbit $O_f(x)$ and the $O(1)$ bound does not depend on x.

Remark

The construction of $\widehat{h}$ uses blow-ups along I_f and $I_{f^{-1}}$. The above estimate ② is similar to Silverman's canonical heights on Wehler K3 surfaces (explained later).

These results are generalized to higher dimensional case by Lee and K.

Theorem (Lee, K)

Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a **regular** *polynomial automorphism of degree $d \geq 2$. Then there exists a canonical height function*

$$\hat{h} : \mathbb{A}^N(\bar{\mathbb{Q}}) \to \mathbb{R}$$

defined in a similar way which satisfies $\widehat{h} \gg\ll h_{\mathrm{Weil}}$ and
$$\hat{h} \circ f + \hat{h} \circ f^{-1} = \left(d + \frac{1}{d}\right) \hat{h}.$$
Further, $\hat{h}$ enjoys the same properties ①② as before.

Remark

To construct $\hat{h}$, Lee uses blow-ups along the indeterminacy sets I_f and $I_{f^{-1}}$. My construction is to introduce local canonical heights and to sum up (as explained in the next page).

❸ Decomposition into the sum of local canonical heights

$f : \mathbb{A}^N \to \mathbb{A}^N$ a **regular** polynomial automorphism of degree $d := \deg(f) \geq 2$ defined over a number field K. Set $d_- := \deg(f^{-1})$.

$x \in \mathbb{A}^N(L)$ for a finite extension L/K, $v \in M_L$: a place

$$G_v^+(x) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^n(x)\|_v,$$
$$G_v^-(x) := \lim_{n\to\infty} \frac{1}{d_-^n} \log^+ \|f^{-n}(x)\|_v.$$

Then

$$\hat{h}(x) := \sum_{v \in M_L} \frac{[L_v : M_v]}{[L : K]} \big(G_v^+(x) + G_v^-(x)\big).$$

Remark

The most difficult part is to show $h_{\mathrm{Weil}} \gg\ll \hat{h}$.

❹ Variation of the canonical height for Hénon maps

Theorem (Ingram)

C a smooth projective curve over a number field K

$f : \mathbb{A}^2 \to \mathbb{A}^2$ a Hénon map of degree $d \geq 2$ defined over $K(C)$

h_C: a Weil height on C corresponding to a divisor of degree 1

$P \in \mathbb{A}^2(K(C))$

Then

$$\hat{h}_t(P_t) = \hat{h}(P)h_C(t) + \varepsilon(t),$$

where $\varepsilon(t) = O(1)$ if $C = \mathbb{P}^1$ and $\varepsilon(t) = O\left(\sqrt{h_C(t)}\right)$ in general.

Thus canonical heights for Hénon maps (and **regular** polynomial automorphisms) enjoy various nice properties.

Canonical heights for other polynomial maps

$f : \mathbb{A}^2 \to \mathbb{A}^2$ a polynomial **map** defined over $\bar{\mathbb{Q}}$

Let $\boldsymbol{\delta} := \boldsymbol{\delta_f} := \lim_{n\to\infty}(\deg f^n)^{1/n}$ be the **(first) dynamical degree** (explained more later). The **topological degree** $\boldsymbol{e} := \boldsymbol{e}_f$ is the number of preimages under f of a general closed point in $\mathbb{A}^2$. By Bézout's theorem, $e \le \delta^2$.

Theorem (Jonsson–Wulcan)

Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial map with $e < \delta$. Then the limit

$$\hat{h}(x) := \lim_{n\to\infty} \frac{1}{\delta^n} h_{\mathrm{Weil}}(f^n(x)),$$

exist for all $x \in \mathbb{A}^2(\bar{\mathbb{Q}})$. One has $\hat{h} \not\equiv 0$ and $\boldsymbol{\hat{h} \circ f = \delta\,\hat{h}}$.

Remark For another case, Jonsson–Reschke construct canonical heights for birational maps of surfaces (explained later).

Part 3 Canonical heights for surface automorphisms

Wehler $K3$ surface

$\boldsymbol{X}$: a complete intersection of general $(1,1)$ and $(2,2)$ hypersurfaces defined over $\bar{\mathbb{Q}}$ in $\mathbb{P}^2 \times \mathbb{P}^2$

$\Longrightarrow X$ is a $K3$ **surface**

The projections $p_i : X \to \mathbb{P}^2$ $(i = 1, 2)$ are double covers, inducing **involutions** $\boldsymbol{\sigma_i} : X \to X$.

$\boldsymbol{D_i} := p_i^*(\text{line}) \in \mathrm{Div}(X)$ $(i = 1, 2)$

Set $E^+ := (1 + \sqrt{3})D_1 - D_2$ and $E^- := -D_1 + (1 + \sqrt{3})D_2$ in $\mathrm{Div}(X)_{\mathbb{R}}$

Let X be a Wehler $K3$ surface with involutions σ_1, σ_2.

Theorem (Silverman 1991)

There exist a unique pair of functions,

$$\boldsymbol{\hat{h}^+}, \boldsymbol{\hat{h}^-} : X(\bar{\mathbb{Q}}) \to \mathbb{R}$$

such that h^+ (resp. h^-) is a height function associated to E^+ (resp. E^-) and such that

$$\begin{aligned} \boldsymbol{\hat{h}^\pm \circ \sigma_1 = (2+\sqrt{3})^\mp \hat{h}^\mp} \ , \\ \boldsymbol{\hat{h}^\pm \circ \sigma_2 = (2+\sqrt{3})^\pm \hat{h}^\mp}. \end{aligned}$$

Further (... continued)

Silverman showed the following properties of canonical heights.

1. $$\boldsymbol{\hat{h}^+(x) = 0} \text{ if and only if } \boldsymbol{\hat{h}^-(x) = 0} \text{ if and only if}$$ $$\boldsymbol{O_{\sigma_1,\sigma_2}(x) := \{\sigma(x) \mid \sigma \in \langle \sigma_1, \sigma_2 \rangle\}} \text{ is a } \textbf{finite set.}$$

2. Take $x \in X(\bar{\mathbb{Q}})$ such that $O_{\sigma_1,\sigma_2}(x)$ is infinite. Then

$$\begin{aligned}\#\{y \in O_{\sigma_1,\sigma_2}(x) \mid h_H(y) \leq T\} \\ &= \epsilon \frac{\log T}{\log(2+\sqrt{3})} - \widehat{h}(O_{\sigma_1,\sigma_2}(x)) + O(1),\end{aligned}$$

where H is any ample divisor on X, $\epsilon = 1$ or 2 (depending on x), $\widehat{h}(O_{\sigma_1,\sigma_2}(x))$ is a quantity defined by the orbit $O_{\sigma_1,\sigma_2}(x)$ and the $O(1)$ bound does not depend on x.

Let X be a Wehler $K3$ surface with involutions σ_1, σ_2.
Put $\boldsymbol{f} := \sigma_1 \circ \sigma_2$.

Then for $E^+ := (1+\sqrt{3})D_1 - D_2$ and $E^- := -D_1 + (1+\sqrt{3})D_2$, one has $\boldsymbol{f^*(E^+) \sim (7+4\sqrt{3})E^+}$ and $\boldsymbol{f^{-1*}(E^-) \sim (7+4\sqrt{3})E^-}$.

So, by Call–Silverman, there exist canonical heights $\boldsymbol{\hat{h}_{E^+}}$ and $\boldsymbol{\hat{h}_{E^-}}$.
Giving $\{\hat{h}^+, \hat{h}^-\}$ is essentially the same as giving $\{\hat{h}_{E^+}, \hat{h}_{E^-}\}$.

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Giving $\{\hat{h}^+, \hat{h}^-\}$ is essentially the same as giving $\{\hat{h}_{E^+}, \hat{h}_{E^-}\}$.

Remark

Since $\hat{h}_{E^+}, \hat{h}_{E^-}$ are Call-Silverman canonical heights, they have properties ③ (decomposition into local canonical heights) and ④ (variation).

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Put $\boldsymbol{f} := \sigma_1 \circ \sigma_2$.

Then for $E^+ := (1+\sqrt{3})D_1 - D_2$ and $E^- := -D_1 + (1+\sqrt{3})D_2$, one has $\boldsymbol{f^*(E^+) \sim (7+4\sqrt{3})E^+}$ and $\boldsymbol{f^{-1*}(E^-) \sim (7+4\sqrt{3})E^-}$.

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Remark

Since $\hat{h}_{E^+}, \hat{h}_{E^-}$ are Call-Silverman canonical heights, they have properties ③ (decomposition into local canonical heights) and ④ (variation).

Remark

$f : X \to X$ is an automorphism of **positive topological entropy** (explained in the next page).

Topological entropy for surface automorphisms

X: a smooth projective surface defined over $\bar{\mathbb{Q}}$

$f : X \to X$: an automorphism

The **(first) dynamical degree** of f is defined by

$$\delta := \max \left\{ |\lambda| \;\middle|\; \begin{array}{c} \lambda \text{ is an eigenvalue of} \\ f^* \otimes \mathbb{C} : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C}) \end{array} \right\}.$$

The **topological entropy** of f equals $\boldsymbol{\log \delta}$ (Gromov, Yomdin).

Remark

1. f has positive topological entropy $\iff \delta > 1$
2. If X has an automorphism of positive topological entropy, then a minimal model of X is either an Abelian surface, K3 surface, Enriques surface, or a rational surface (Cantat).

Surface automorphism of positive topological entropy

X: a smooth projective surface defined over $\bar{\mathbb{Q}}$

$f : X \to X$: an automorphism of positive topological entropy $\log \delta$.

An irreducible curve C is **periodic** if $f^n(C) = C$ (as a set) for some $n \geq 1$.

Theorem (K.)

- *There are at most finitely many irreducible periodic curves.*
- *There exist nef divisors E^+ and $E^- \in \mathrm{Div}(X)_{\mathbb{R}}$ such that $f^*(E^+) \sim \delta E^+$ and $f^{-1*}(E^-) \sim \delta E^-$. (They are unique up to scales.) Further, $E^+ + E^-$ is nef and big.*
- *By the above, we have Call–Silverman canonical heights $\hat{h}_{E^+}$ and $\hat{h}_{E^-}$. We set $\boldsymbol{\hat{h} := \hat{h}_{E^+} + \hat{h}_{E^-}}$, which is a height function associated to $E^+ + E^-$.*

  Further (... continued)

1. $\boldsymbol{\hat{h}(x) = 0}$ if and only if "x lies on a **periodic curve** or x is **periodic**."

   In particular, the set $\mathbf{Per}(f, \bar{\mathbb{Q}}) \setminus$ **periodic curves** is a set of bounded height.

2. Take $x \in X(\bar{\mathbb{Q}})$ such that x does not lies on a periodic curve and that x is not periodic. Then

$$\#\{y \in O_f(x) \mid h_H(y) \leq T\} = 2\frac{\log T}{\log \delta} - \widehat{h}(O_f(x)) + O(1),$$

   where $O_f(x) = \{f^n(x) \mid n \in \mathbb{Z}\}$, H is any ample divisor on X, $\widehat{h}(O_f(x))$ is a quantity defined by the orbit $O_f(x)$ and the $O(1)$ bound does not depend on x.

Birational surface maps

Recently Jonsson–Reschke show the existence of canonical heights for birational surface maps.

Theorem (Jonsson–Reschke)

Let X be a smooth projective surface defined over $\bar{\mathbb{Q}}$, and $f : X \dashrightarrow X$ a birational selfmap. Assume that the first dynamical degree $\delta > 1$. Then, up to birational conjugacy, the limit

$$\hat{h}^+(x) := \lim_{n\to\infty} \frac{1}{\delta^n} h_{E^+}(f^n(x)),$$

exists and non-negative for all $x \in X(\bar{\mathbb{Q}})$ with well-defined forward orbit.

A remark

Let $d > 1$, and

$\{a_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $a_n \geq 1$.

Assume that $\displaystyle\lim_{n\to\infty} \frac{1}{d^n} a_n$ exists, and not zero.
Then $\displaystyle\lim_{n\to\infty} a_n^{1/n}$ exists, and equals to d.

The converse does not hold in general.

Part 4 Arithmetic degrees

A remark

Let $d > 1$, and

$\{a_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $a_n \geq 1$.

Assume that $\displaystyle \lim_{n\to\infty} \frac{1}{d^n} a_n$ exists, and not zero.

Then $\displaystyle \lim_{n\to\infty} a_n^{1/n}$ exists, and equals to d.

The converse does not hold in general.

This suggests that "$\displaystyle \lim_{n\to\infty} h(f^n(x))^{1/n}$" might exist even if the canonical height "$\displaystyle \lim_{n\to\infty} \frac{1}{d^n} h(f^n(x))$" does not exist.

$\boldsymbol{X}$: a smooth projective variety over $\bar{\mathbb{Q}}$

$\boldsymbol{f : X \dashrightarrow X}$: a dominant rational map

We fix (any) ample divisor H on X and

a Weil height $h_H : X(\bar{\mathbb{Q}}) \to [1, \infty)$ associated to H

$\boldsymbol{X(\bar{\mathbb{Q}})_f} := \{x \in X(\bar{\mathbb{Q}}) \mid f^n(x) \text{ is well-defined for all } n \geq 0\}$

Definition (arithmetic degree, Silverman)

Let $x \in X(\bar{\mathbb{Q}})_f$. If the limit $\lim_{n\to\infty} h_H\left(f^n(x)\right)^{1/n}$ exists, then we set

$$\boldsymbol{\alpha_f(x)} := \lim_{n\to\infty} h_H\left(f^n(x)\right)^{1/n}.$$

and call the **arithmetic degree** of x for f.

In general, we set $\boldsymbol{\overline{\alpha}_f(x)} := \limsup_{n\to\infty} h_H\left(f^n(x)\right)^{1/n}$.

Remark

The arithmetic degree $\alpha_f(x) := \lim_{n\to\infty} h_H\left(f^n(x)\right)^{1/n}$ (if exists) measures the "**size**" of the forward orbit $O_f(x)$ of x:

$$\#\{y \in O_f(x) \mid h_H(y) \le T\} \sim \frac{\log T}{\log \alpha_f(x)} \quad \text{as } T \to \infty.$$

Remark

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$$\#\{y \in O_f(x) \mid h_H(y) \le T\} \sim \frac{\log T}{\log \alpha_f(x)} \quad \text{as } T \to \infty.$$

First dynamical degree

X a smooth projective variety over $\bar{\mathbb{Q}}$

$f : X \dashrightarrow X$ a dominant rational map

$\mathbf{NS}(\boldsymbol{X}) := \mathrm{Div}(X)/(\text{algebraic equivalence})$

Néron-Severi group of X

Then $\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{NS}(X) \otimes \mathbb{R}$ (resp. $\mathrm{NS}(X)_{\mathbb{C}} := \mathrm{NS}(X) \otimes \mathbb{C}$)

a finite dimensional $\mathbb{R}$-vector (resp. $\mathbb{C}$-vector) space.

Definition (dynamical degree of f, one of equivalent definitions)

$f^n : X \dashrightarrow X$ induces a linear transformation f^{n*} on $\mathrm{NS}(X)$.

$$\boldsymbol{d(f^n)} := \max \left\{ |\lambda| \;\middle|\; \begin{array}{c} \lambda \text{ is an eigenvalue of} \\ f^{n*} \otimes \mathbb{C} : \mathrm{NS}(X)_{\mathbb{C}} \to \mathrm{NS}(X)_{\mathbb{C}} \end{array} \right\}$$

Then the **(first) dynamical degree** of f is defined by

$$\boldsymbol{\delta} := \boldsymbol{\delta_f} := \lim_{n\to\infty} d(f^n)^{1/n}$$

Remark

Dynamical degrees have been extensively studied in complex dynamical systems and integrable systems.

Relation between the dynamical degree and the arithmetic degree

X: a smooth projective variety over $\bar{\mathbb{Q}}$

$f : X \dashrightarrow X$: a dominant rational map

δ_f: dynamical degree of f

$\overline{\alpha}_f(x) := \limsup_{n\to\infty} h_H\left(f^n(x)\right)^{1/n}$

arithmetic degree of any $x \in X(\bar{\mathbb{Q}})_f$ for f

Theorem (Silverman ($X = \mathbb{P}^N$); K.–Silverman (X general))

In the above setting, one has

$$\overline{\alpha}_f(x) \leq \delta_f.$$

Monomial maps

$f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is a **monomial map**
if it is a rational extension of $f : \mathbb{G}_m^N \to \mathbb{G}_m^N$,
$f = (X_1^{a_{11}} \cdots X_N^{a_{1N}}, \ldots, X_1^{a_{N1}} \cdots X_N^{a_{NN}})$ for some $A = (a_{ij}) \in M_N(\mathbb{Z})$.

Theorem (Silverman 2014)

Let $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a monomial map such that A is diagonalizable.

1. *The limit defining* $\boldsymbol{\alpha_f(x) = \lim_{n\to\infty} h_H\left(f^n(x)\right)^{1/n}}$ *exists (and independent of the choice of an ample divisor H).*
2. $\alpha_f(x)$ *is an* **algebraic integer** *for any* $\mathbb{G}_m^N(\bar{\mathbb{Q}})$.
3. $\{\alpha_f(x) \mid x \in \mathbb{G}_m^N(\bar{\mathbb{Q}})\}$ *is a* **finite set**.
4. *Let $x \in \mathbb{G}_m^N(\bar{\mathbb{Q}})$. Then, if $O_f^+(x)$ is Zariski dense in $\mathbb{P}^N$, then* $\alpha_f(x) = \delta_f$.

Conjecture (Silverman)

X: a smooth projective variety over $\bar{\mathbb{Q}}$

$f : X \dashrightarrow X$: a dominant rational map

1. The limit defining $\boldsymbol{\alpha_f(x) = \lim_{n\to\infty} h_H\left(f^n(x)\right)^{1/n}}$ exists.
2. $\alpha_f(x)$ is an **algebraic integer** for any $X(\bar{\mathbb{Q}})_f$.
3. $\{\alpha_f(x) \mid x \in X(\bar{\mathbb{Q}})_f\}$ is a **finite set**.
4. Let $x \in X(\bar{\mathbb{Q}})_f$. Then, if $O_f^+(x)$ is Zariski dense in X, then $\alpha_f(x) = \delta_f$.

Remark

The conjecture ② corresponds to Bellon–Viallet's conjecture, which asks whether the dynamical degree δ_f is an algebraic integer.

The conjecture ④ seems the most difficult.

Conjectures for morphisms

X: a smooth projective variety over $\bar{\mathbb{Q}}$

$f : X \to X$: a **morphism**

Theorem (K.–Silverman)

Conjectures ①②③ hold true for morphisms:

1. *The limit defining* $\alpha_f(x) = \lim_{n\to\infty} h_H\left(f^n(x)\right)^{1/n}$ *exists.*
2. $\alpha_f(x)$ *is an algebraic integer for any* $X(\bar{\mathbb{Q}})$*.*
3. $\{\alpha_f(x) \mid x \in X(\bar{\mathbb{Q}})\}$ *is a finite set.*

Remark

To prove the above statements,
we study the action of f^* on $\mathrm{Div}(X)_{\mathbb{C}}$, and construct nice height functions related to f.

Jordan Block Canonical Heights

X: a projective variety over $\bar{\mathbb{Q}}$

$f : X \to X$: a **morphism**

$\boldsymbol{\lambda} \in \mathbb{C}$ with $|\lambda| > 1$,

Assume that there exist $\boldsymbol{D_0, D_1, D_2 \ldots, \in \mathbf{Div}(X) \otimes \mathbb{C}}$ be such that

$$\begin{aligned} f^*D_0 &\sim \lambda D_0, \\ f^*D_1 &\sim D_0 + \lambda D_1, \\ f^*D_2 &\sim \qquad\quad D_1 + \lambda D_2, \\ &\vdots \qquad\qquad \ddots \qquad \ddots \end{aligned}$$

Then, for $k = 0$, the limit $\boldsymbol{\hat{h}_{D_0}(x) := \lim_{n\to\infty} \frac{1}{\lambda^n} h_{D_0}(f^n(x))}$ converges (Call–Silverman), but, for $k \geq 1$, the limit

$$\lim_{n\to\infty} \frac{1}{\lambda^n} h_{D_k}(f^n(x))$$

does **not** converge in general.

Theorem (Canonical heights for Jordan blocks)

There exist unique height function $\boldsymbol{\hat{h}_{D_k}}$ associated to D_k for all $k = 0, 1, 2, \ldots$,

$$\boldsymbol{\hat{h}_{D_k} : X(\bar{\mathbb{Q}}) \to \mathbb{R},}$$

satisfying $\boldsymbol{\hat{h}_{D_k}(f(x)) = \lambda \hat{h}_{D_k}(x) + \hat{h}_{D_{k-1}}(x)}$ for all $k = 0, 1, 2, \ldots$.

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Remark

Concretely, recursively for $k = 0, 1, 2, \ldots,$ we have

$$\hat{h}_{D_k}(x) := \lim_{n\to\infty}\left(\frac{1}{\lambda^n} h_{D_k}\left(f^n(x)\right) - \sum_{i=1}^{k}\binom{n}{i}\frac{1}{\lambda^i}\hat{h}_{D_{k-i}}(x)\right).$$

Using canonical heights for Jordan blocks, we see that conjectures ①②③ hold true for **morphisms**.

Remark

If $|\lambda| > \sqrt{\delta_f}$, then a similar statement is true with the linear equivalences replaced by **numerical equivalences**.
However, the condition $\hat{h}_{D_k} = h_{D_k} + O(1)$ is replaced by $\hat{h}_{D_k} = h_{D_k} + O(\sqrt{h_H})$. (Here H is an ample divisor on X.)

Indeed, using $\overline{\alpha}_f(x) \leq \delta_f$, one can show that if $f : X \to X$ is a morphism, and $D \in \mathrm{Div}(X)_{\mathbb{R}}$ is a divisor such that $f^*[D] = \lambda[D]$ in $\mathrm{NS}(X)_{\mathbb{R}}$ for some $\lambda > \sqrt{\delta_f}$, then the limit

$$\hat{h}_D(x) := \lim_{n\to\infty} \frac{1}{\lambda^n} h_D(f^n(x))$$

exists (Here h_D is a Weil height associated to D).

Conjecture ④ for isogenies of abelian varieties

A: an abelian variety over $\bar{\mathbb{Q}}$

$f : A \to A$ an isogeny (i.e., a surjective group endomorphism)

In this case, in addition to Conjectures ①②③, Conjecture ④ holds true.

Theorem (K.–Silverman)

Let $x \in A(\bar{\mathbb{Q}})$. If the forward orbit $O_f^+(x)$ is Zariski dense in A, then $\alpha_f(x) = \delta_f$.

Remark

For an isogeny in general, one can find $D \neq 0 \in \mathrm{NS}(A)_{\mathbb{R}}$ such that $f^*D = \delta_f D$. But often D is **not** ample, and is only **nef**.

Example

E: elliptic curve over $\bar{\mathbb{Q}}$ without CM

$\boldsymbol{A := E \times E}$

Automorphism of A:

$$\boldsymbol{f} : A \to A,\ (x, y) \mapsto ([a]\,x + [b]\,y, [c]\,x + [d]\,y),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ with $a + d > 2$.

Let $\mu > 1$ be a root of the characteristic polynomial $t^2 - (a + d)t + 1$.

Then

$^\exists \boldsymbol{D} \in \mathrm{NS}(A)_{\mathbb{R}}$ such that D is **nef** and symmetric, and

$f^* D = \mu^2\, D \quad$ in $\mathrm{NS}(A)_{\mathbb{R}}$.

In this case, D is not ample.

Theorem (nef canonical height theorem)

Let A be an abelian variety over $\bar{\mathbb{Q}}$, and
$D \in \mathrm{Div}(A)_{\mathbb{R}}$ a **nef** and symmetric $\mathbb{R}$-divisor such that $D \not\sim_{lin} 0$.
Let $\hat{h}_D$ be the Néron-Tate height. Then

1. $\hat{h}_D(x) \geq 0$ for any $x \in A(\bar{\mathbb{Q}})$.
2. There exists a subabelian variety $B_D \subsetneq A$ over $\bar{\mathbb{Q}}$ such that

$$\{x \in A(\bar{\mathbb{Q}}) \mid \hat{h}_D(x) = 0\} = B_D(\bar{\mathbb{Q}}) + A(\bar{\mathbb{Q}})_{tor}.$$

Remark

If D is **ample**, then $\{x \in A(\bar{\mathbb{Q}}) \mid \hat{h}_D(x) = 0\} = A(\bar{\mathbb{Q}})_{tor}$.

Sketch of proof

For an isogeny $f : A \to A$ one can find a nef and symmetric $D \neq 0 \in \mathrm{NS}(A)_{\mathbb{R}}$ such that $f^*D = \delta_f D$. Suppose $\alpha_f(x) < \delta_f$. Then one can show that $\hat{h}_D(x) = 0$. Using the nef canonical height theorem, one can show that $O_f^+(x)$ is included in a proper subvariety of A.