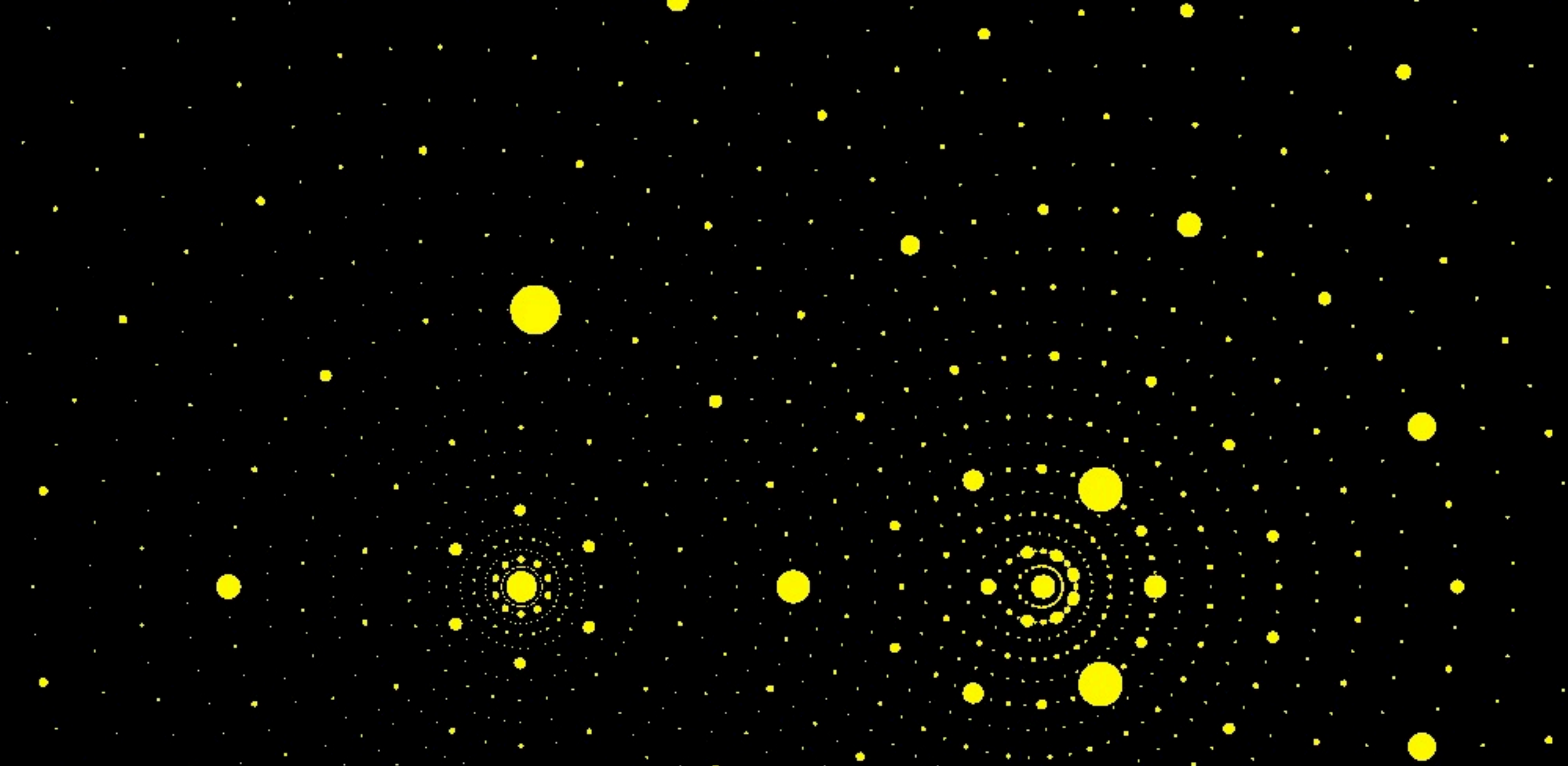
Variation of canonical height, illustrated

Laura DeMarco
Northwestern University



Theorem I.0.3. (Silverman, VCH I, 1992)

$P = (0, 0)$

$E = \{y^2 + Txy + Ty = x^3 + 2Tx^3\}$

$$\hat{h}_{E_t}(P_t) = \frac{1}{15}\log t + \frac{2}{25}\log 2 + \frac{2}{25}\frac{(\log 2)^2}{\log(t^5/2)} + O(t^{-1}) \text{ for } t \in \mathbb{Z},\ t \to \infty$$

Variation of canonical height, illustrated

- Brief overview:
  families of elliptic curves
- Connections with dynamics
- pictures
- Rationality of canonical heights?
  (work in progress with Dragos Ghioca)

E = elliptic curve / number field K

$$y^2 = x^3 + Ax + B \qquad A, B \in K$$

Néron-Tate (canonical) height function

$$\hat{h}_E(P) = \lim_{n\to\infty} \frac{1}{4^n} h_{\mathrm{Weil}}([2^n P]_x)$$

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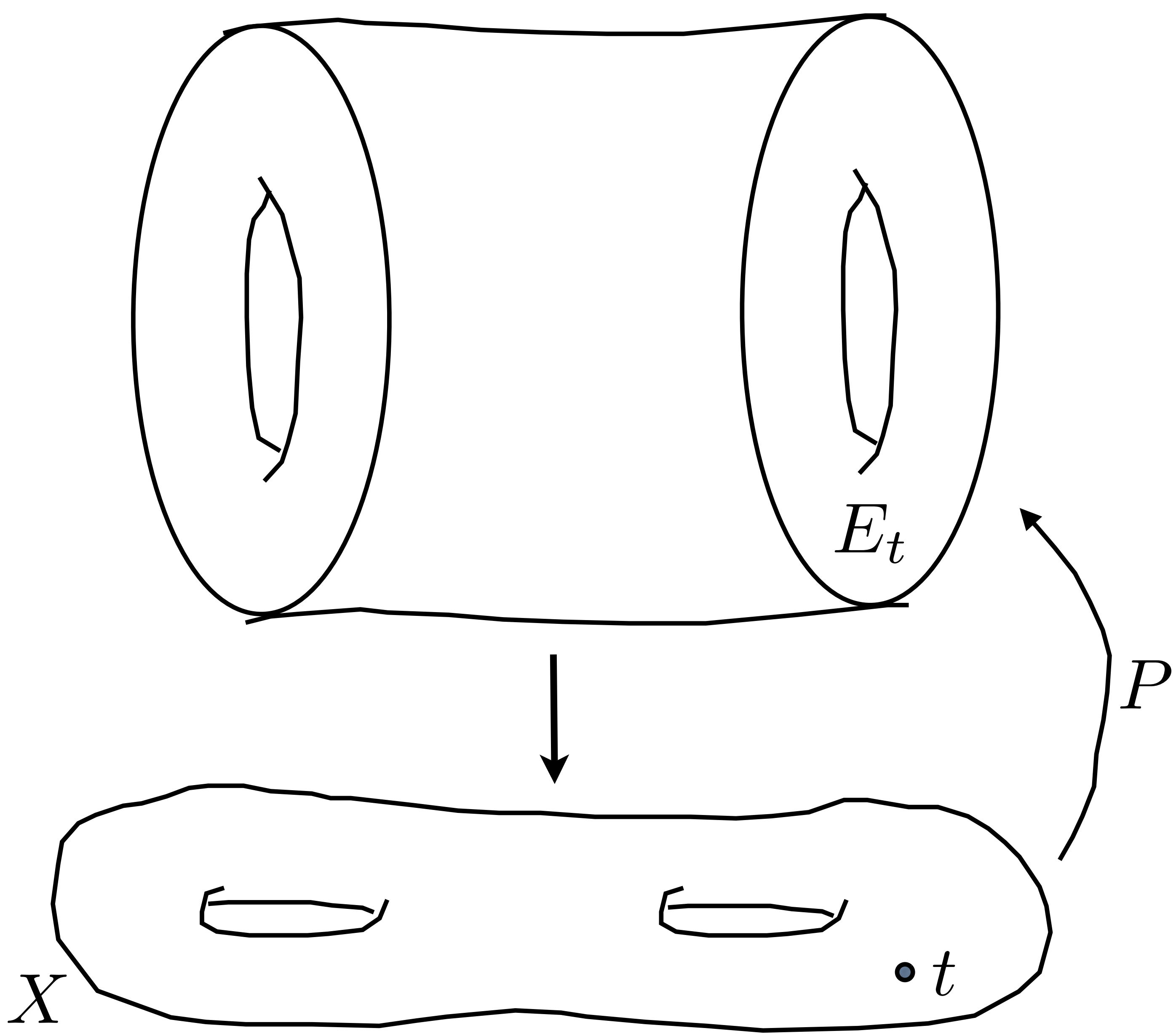
$$P \in E(k)$$

Study $\hat{h}_{E_t}(P_t)$ for $t \in X(\bar{K})$



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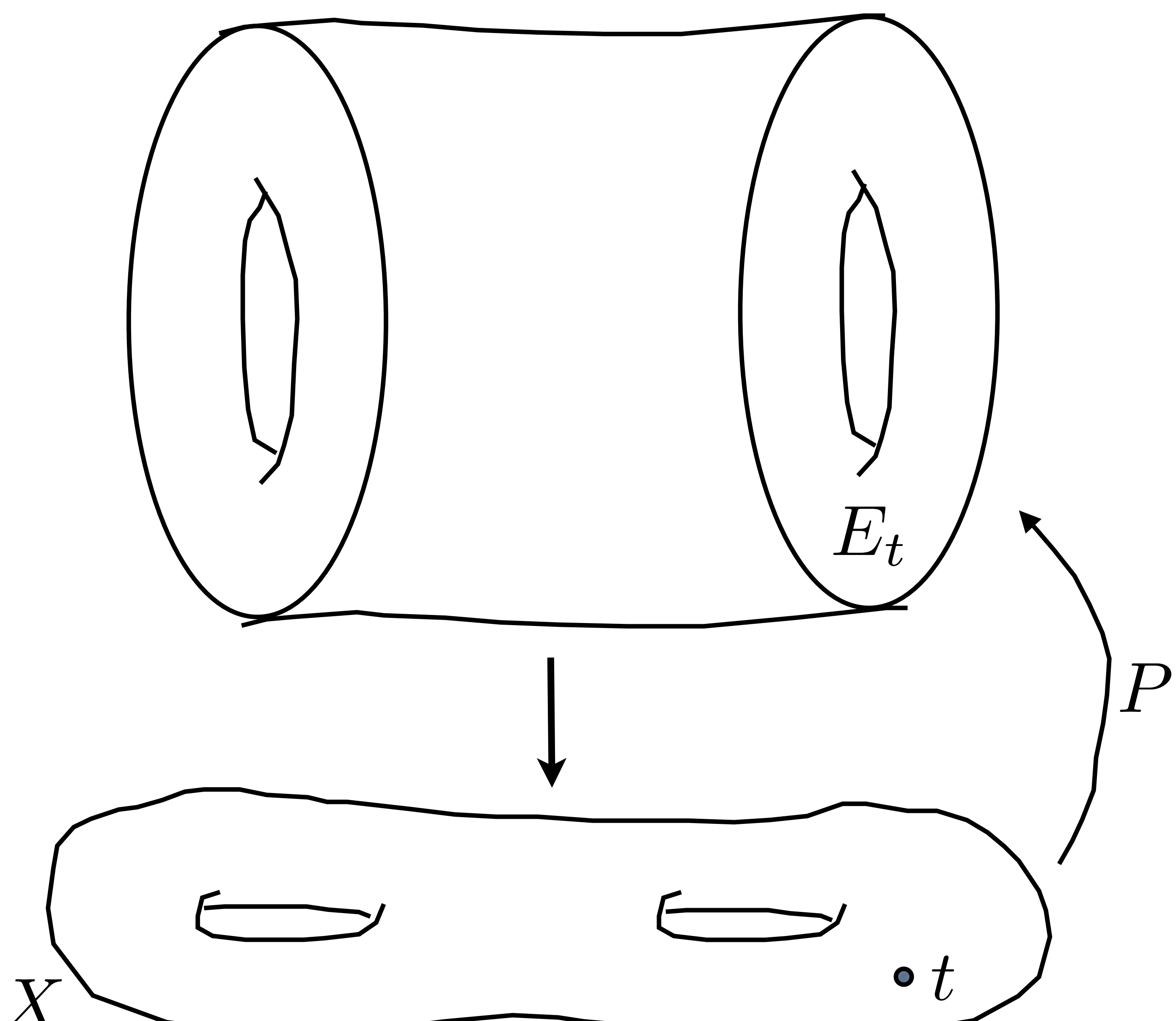

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$$\lim_{h_X(t)\to\infty} \frac{\hat{h}_{E_t}(P_t)}{h_X(t)} = \hat{h}_E(P)$$

Theorem. (Tate, 1983)

$$\hat{h}_{E_t}(P_t) = h_{X,D_P}(t) + O(1)$$

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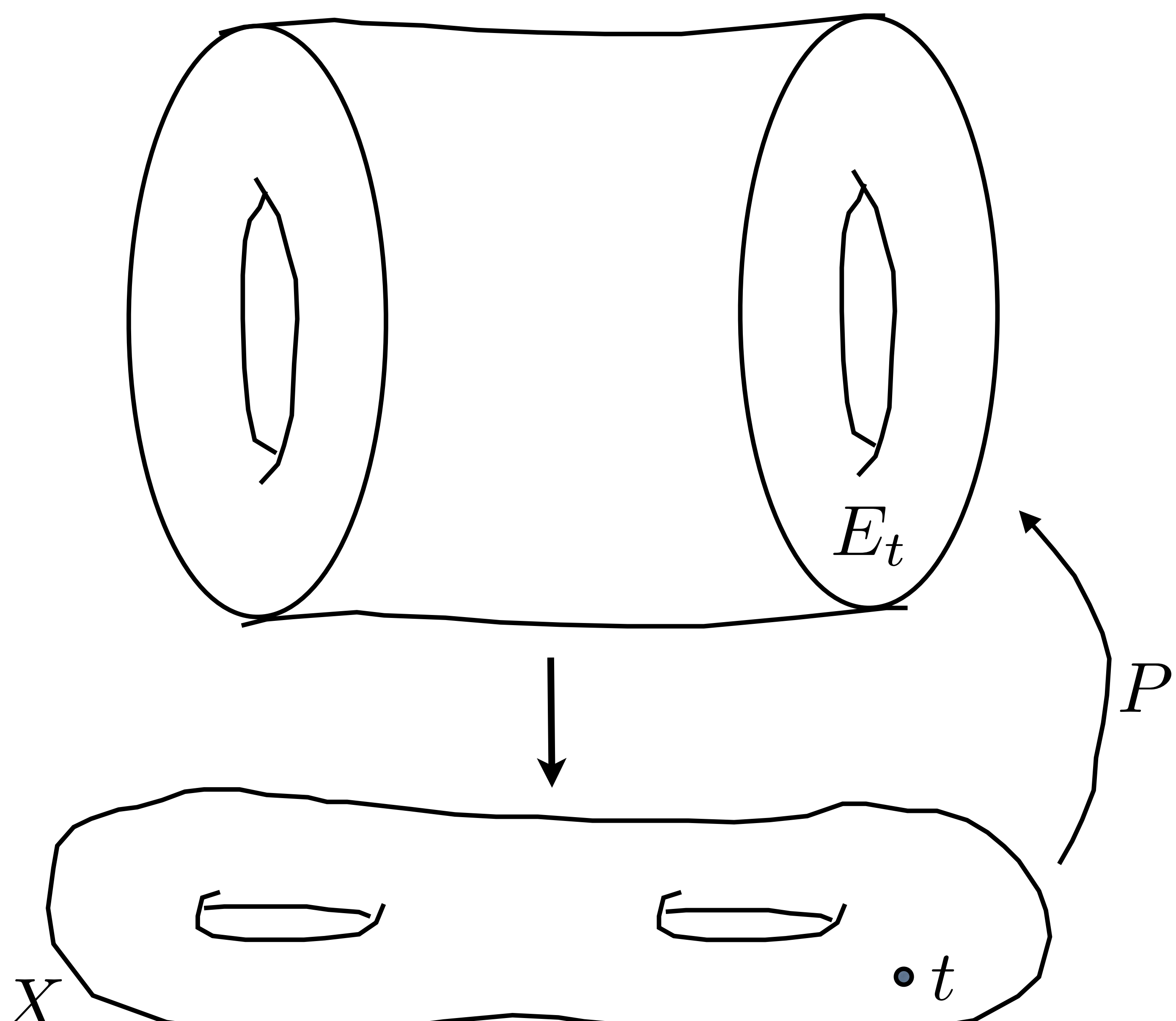

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Silverman's VCH I, II, III, 1992-1994

The components in the local decomposition

$$\hat{h}_{E_t}(P_t) = \sum_{v \in M_K} \hat{\lambda}_{E_t,v}(P_t)$$

satisfy

(1) $\hat{\lambda}_{E_t,v}(P_t) = \hat{\lambda}_{E,t_0}(P) \log|u(t)|_v$ + continuous correction term

(2) correction term $\equiv 0$ for all but finitely many $v \in M_K$

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"I've always thought it was intriguing that the difference h(P_t) - h(P)h(t) is [so well behaved]. On the other hand, I've never found a good application."

- Joe Silverman, July 20, 2015

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for t near $t_0 \in X(\bar{K})$.

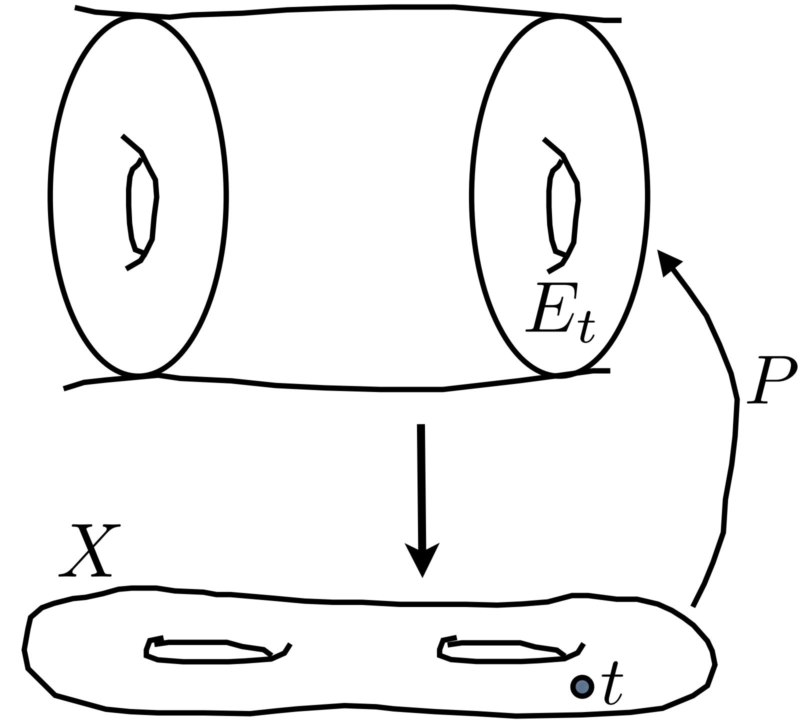
$\Longrightarrow$ $\hat{h}_{E_t}(P_t)$ defines a "good height function" on $X(\bar{K})$
i.e., a continuous potential function for an adelic measure $\mu = \{\mu_v\}$
(or an adelic metrized line bundle, in sense of Zhang, 1995)

$\Longrightarrow$ we are set up to study the distribution of "small" points on X
(e.g. Baker--Rumely, Chambert-Loir, Favre--Rivera-Letelier 2006, Yuan 2008)

$K =$ number field

$E =$ elliptic curve / function field $k = K(X)$

$P \in E(k)$



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$$\mu_v = \Delta(\text{correction term})$$

What are these measures on X?

(strictly speaking, the measures live on the Berkovich analytification of X)

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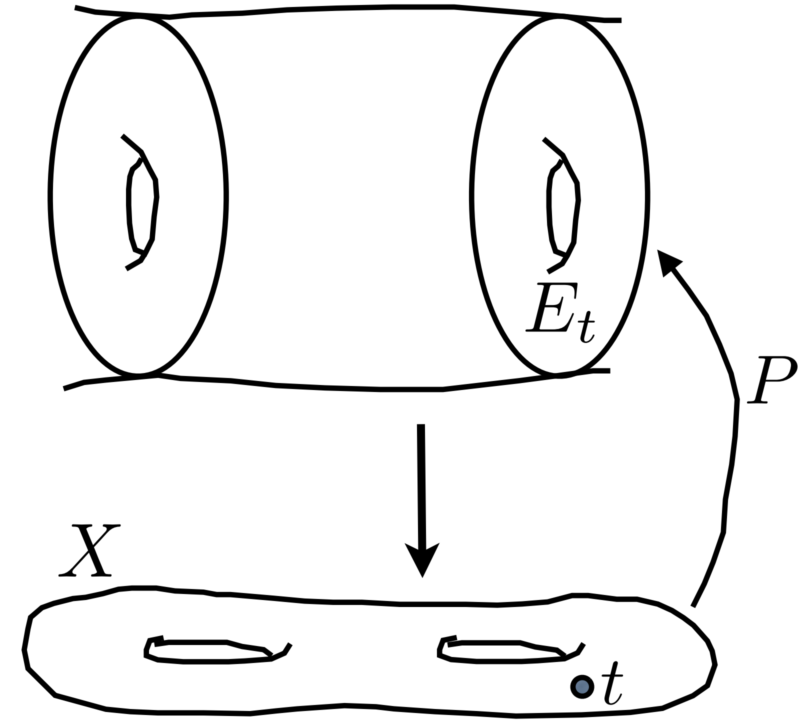

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The measure is a pull-back of the Haar measures on the elliptic curves. This is a special case of the **dynamical bifurcation measure** and the correction term governs the "intensity" of the bifurcation.

Call-Silverman canonical height (1994)

$f : \mathbb{P}^1 \to \mathbb{P}^1$

$\hat{h}_f : \mathbb{P}^1(\bar{K}) \to \mathbb{R}$
determined uniquely by two properties: $\begin{cases} \hat{h}_f(f(z)) = (\deg f)\hat{h}_f(z) \\ \hat{h}_f(z) = h(z) + O(1) \end{cases}$

$$\hat{h}_f(z) = \lim_{n\to\infty} \frac{1}{(\deg f)^n} h(f^n(z))$$

$$= \sum_{v \in M_K} \hat{\lambda}_{f,v}(z)$$

Study variation $\hat{h}_{f_t}(P_t)$ for $t \in X$, in families $\{f_t\}$.
Take the Laplacian Δ of the local heights, as functions of t.

The variation of the canonical height -- at the archimedean place -- quantifies bifurcations in a traditional dynamical sense.

Example: degree 2 polynomials

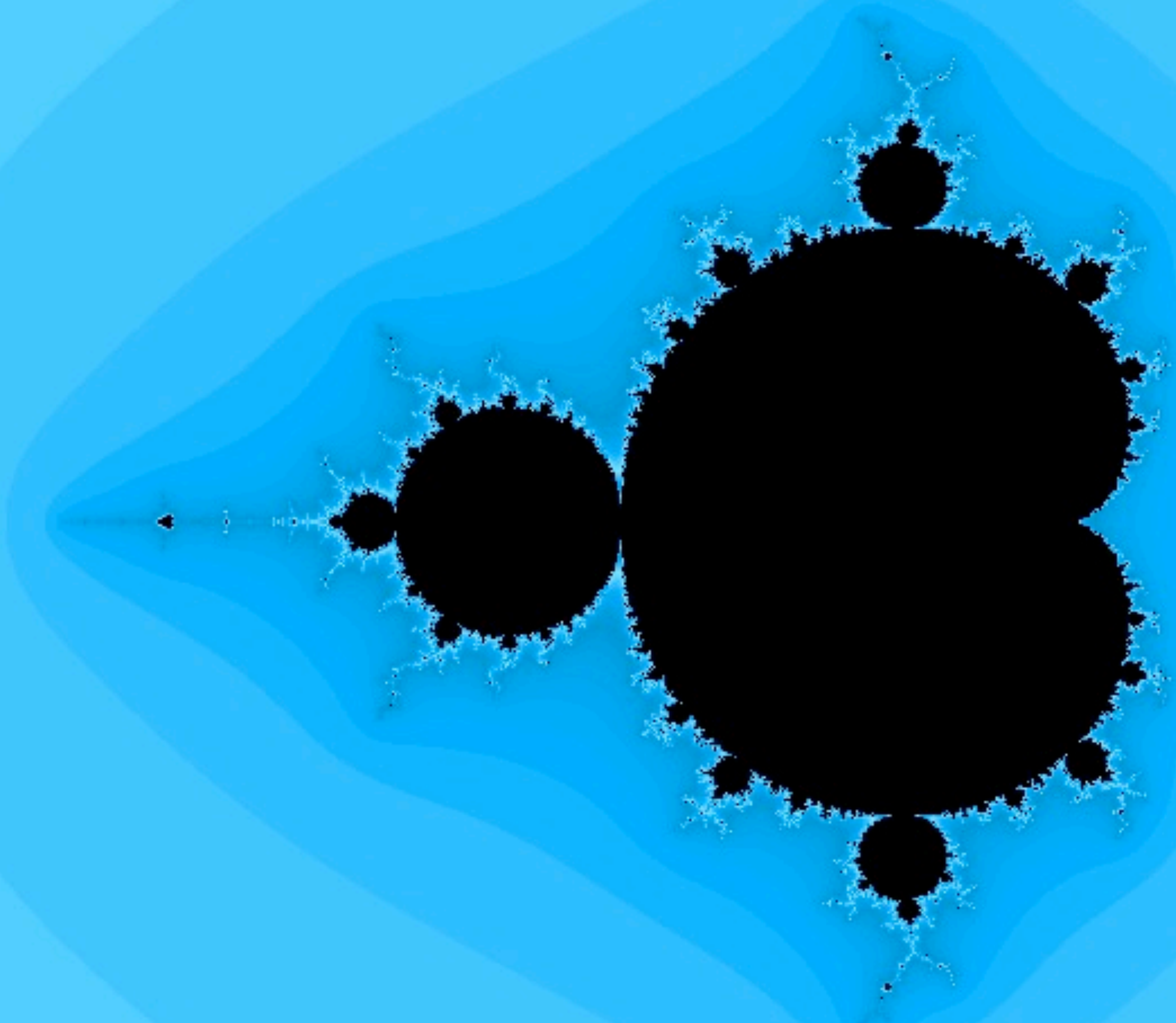
$$f_t(z) = z^2 + t \qquad t \in \mathbb{C}$$

$$P = 0$$

$$\hat{\lambda}_{f_t, v=\infty}(P_t) = \frac{1}{2} \log |t| + \text{correction term}$$

for $|t|$ large

The Mandelbrot set



Bifurcation measure μ_P is harmonic measure on $\partial\mathcal{M}$

(Douady-Hubbard, Sibony 1981, Mañé-Sad-Sullivan 1983)

Example: degree 2 polynomials

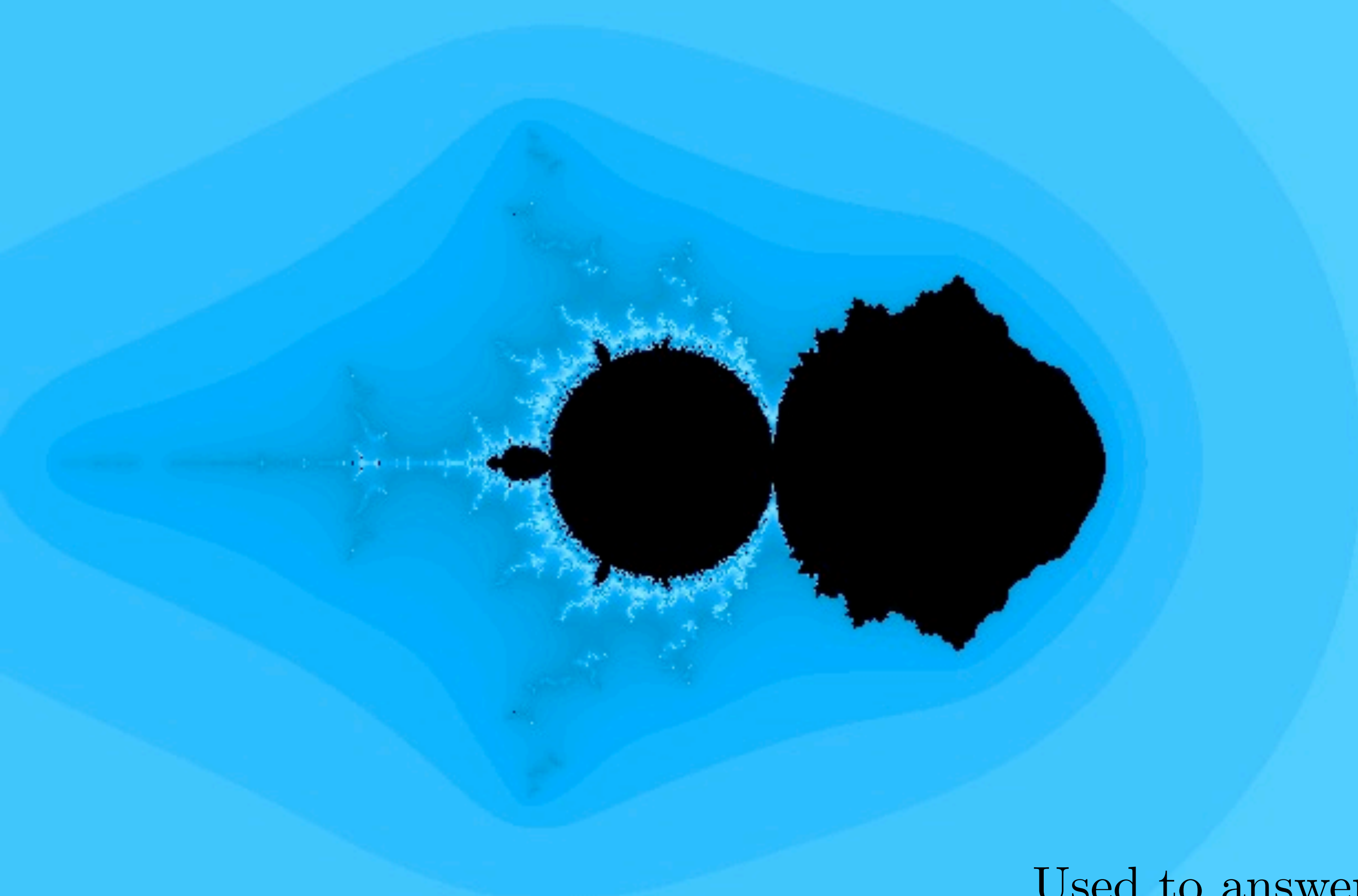
$$f_t(z) = z^2 + t \qquad t \in \mathbb{C}$$

$$P = 1$$

$$\hat{\lambda}_{f_t, v=\infty}(P_t) = \frac{1}{2}\log|t| + \text{correction term}$$

for $|t|$ large

A Mandelbrot-like set



Bifurcation measure μ_P is harmonic measure on $\partial\mathcal{M}$

Used to answer an "unlikely intersections" question posed by Zannier: there are only finitely many t for which both 0 and 1 have finite orbit for f_t.

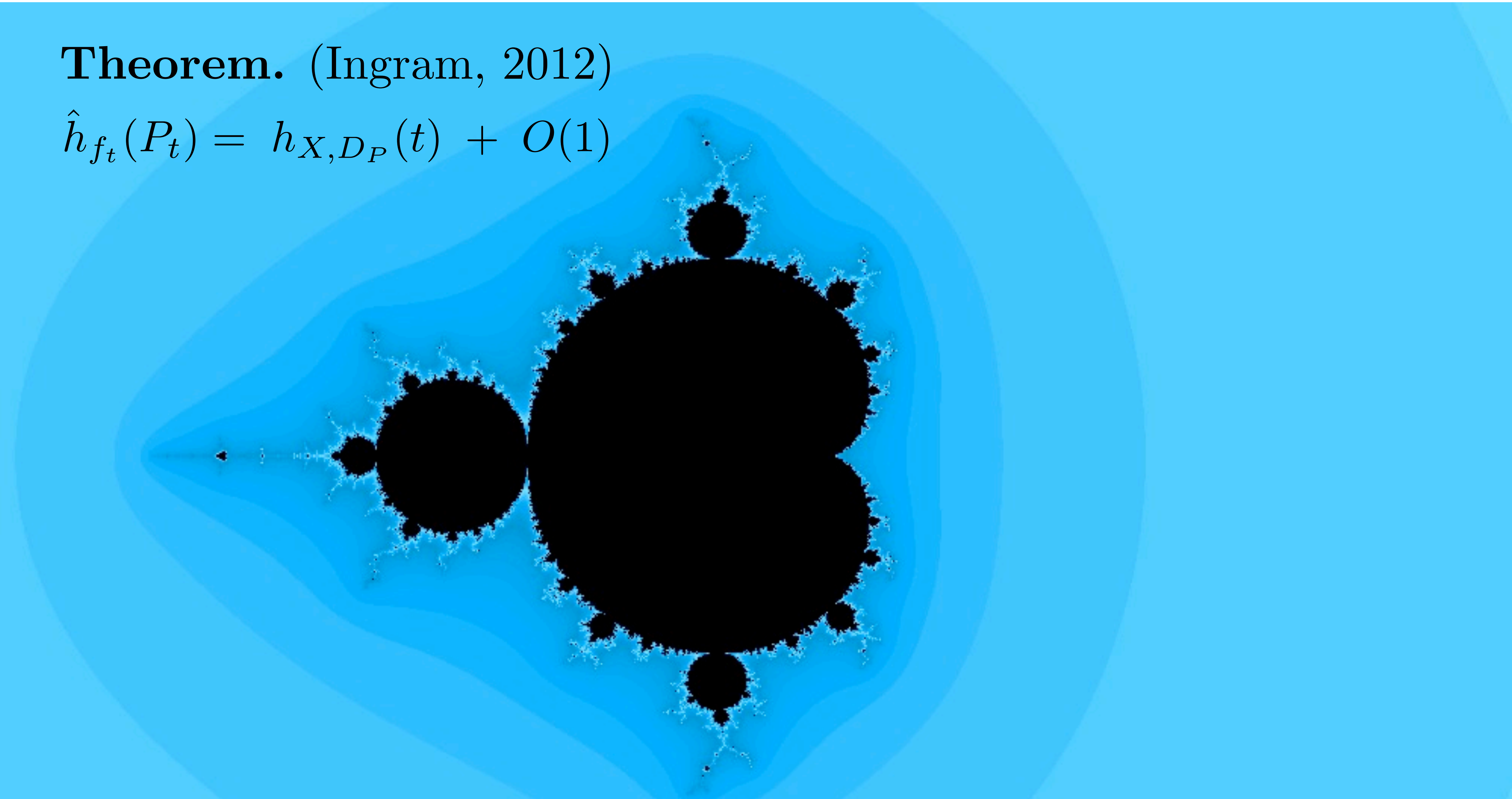
(Baker-D. 2011)

In these examples, the measures are compactly supported (away from point of bad reduction at infinity). So the "correction terms" will be nice harmonic functions near infinity.

For general families of **polynomials**, height functions and measures depend only on rates of escape to infinity. Ingram proved the analog of Tate's 1983 result:

Theorem. (Ingram, 2012)

$$\hat{h}_{f_t}(P_t) = h_{X,D_P}(t) + O(1)$$



Example, in the context of Silverman's VCH I,II, III

$$E_t = \{y^2 = x(x-1)(x-t)\}$$

$$P = (a, \sqrt{a(a-1)(a-t)}) \qquad a \in \mathbb{Q}(t)$$

(1) $\hat{\lambda}_{E_t,v}(P_t) = \hat{\lambda}_{E,t_0}(P) \log |u(t)|_v +$ continuous correction term

(2) correction term $\equiv 0$ for all but finitely many $v \in M_K$

$$\mu_v = \Delta(\text{correction term})$$

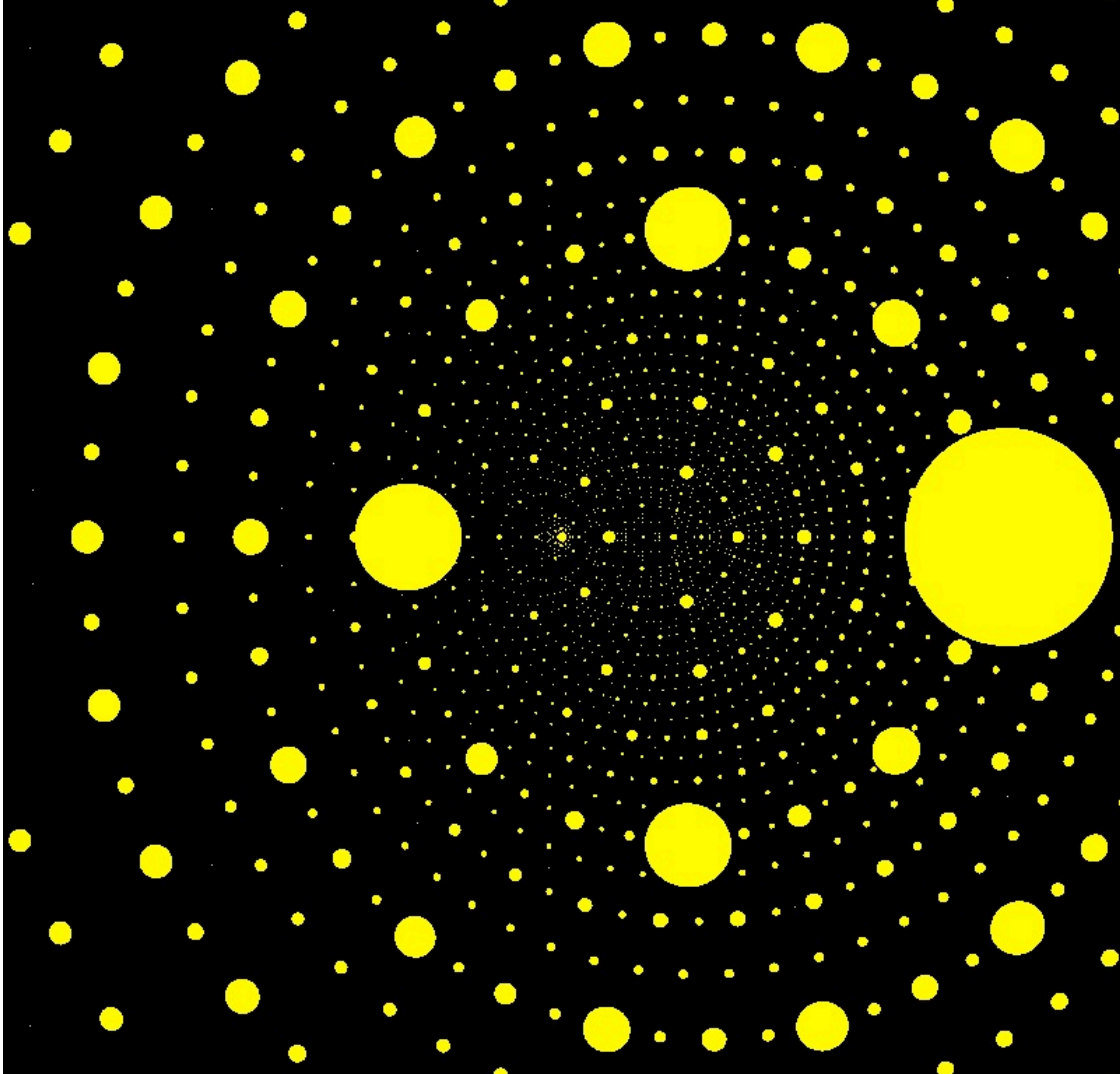
Fact 1. The parameters $t \in X$ where P_t is torsion on E_t are equidistributed with respect to these measures $\mu_{P,v}$.

Fact 2. The measures $\{\mu_{P_v}\}$ coincide with $\{\mu_{Q_v}\}$ if and only if the points P and Q are linearly related on E.
This can be seen already at the archimedean place.

(D.-Wang-Ye, 2015) building on the results of (Masser-Zannier, 2008, 2010, 2012)

$a = 2$

Plot: parameters t where a is the x-coordinate of a torsion point on E_t, of order 2^n with $n < 8$.



$-3 < \operatorname{Re} t < 5$

$-4 < \operatorname{Im} t < 4$

$a = 2$

Plot: parameters t where a is the x-coordinate of a torsion point on E_t, of order 2^n with $n < 10$.

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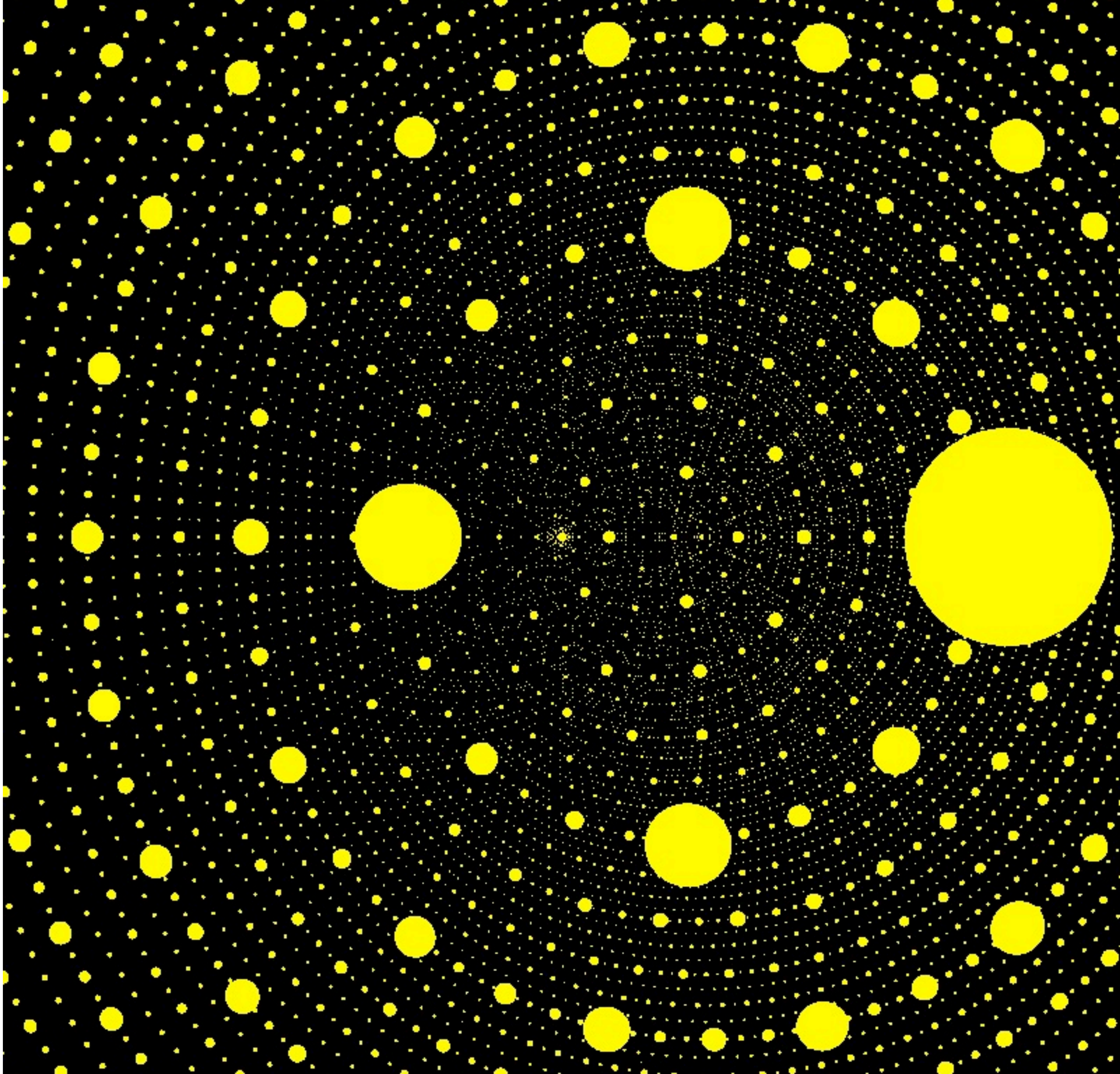

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Plot: parameters t where a is the x-coordinate of a torsion point on E_t, of order 2^n with $n < 15$.

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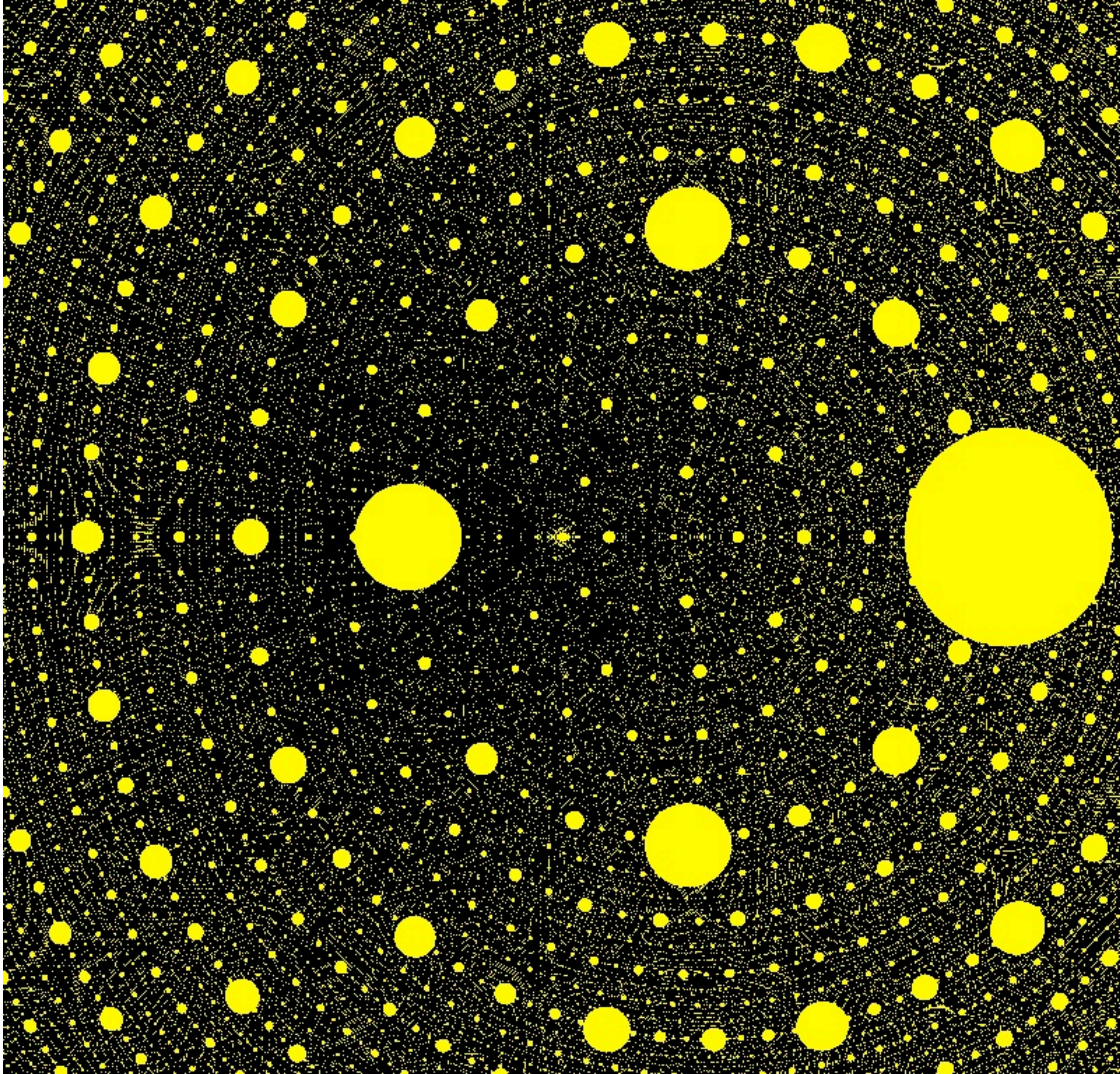

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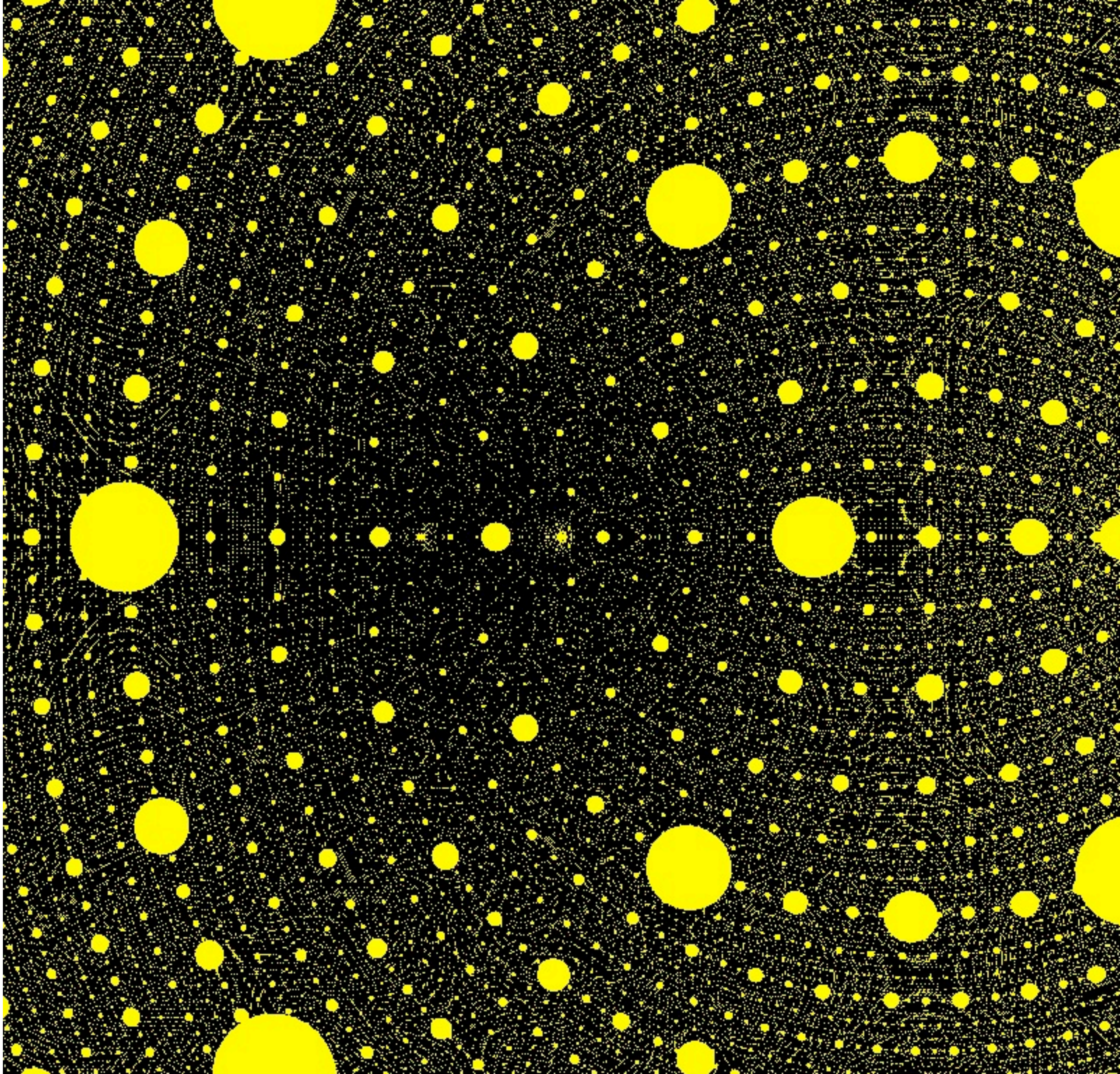
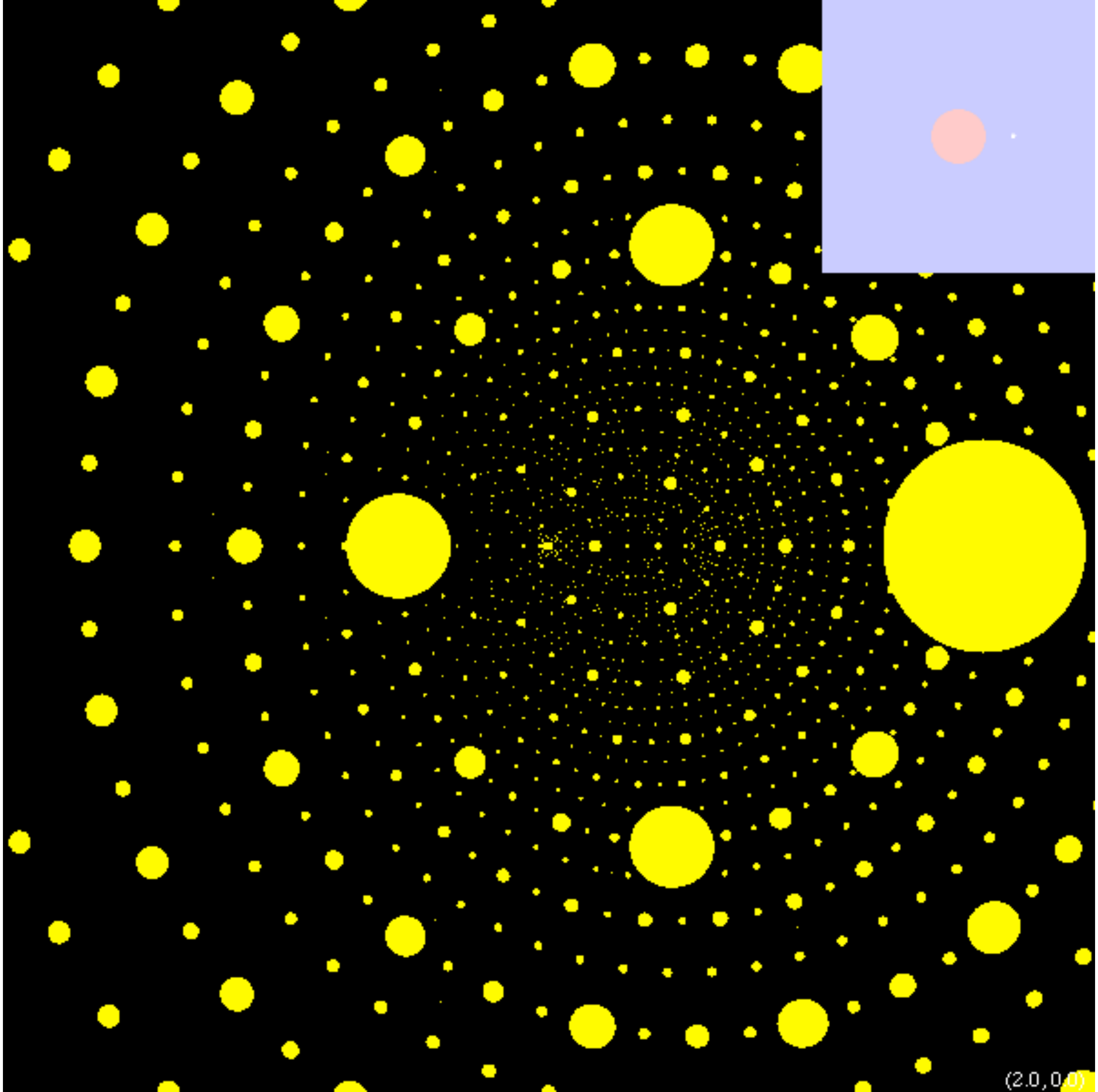
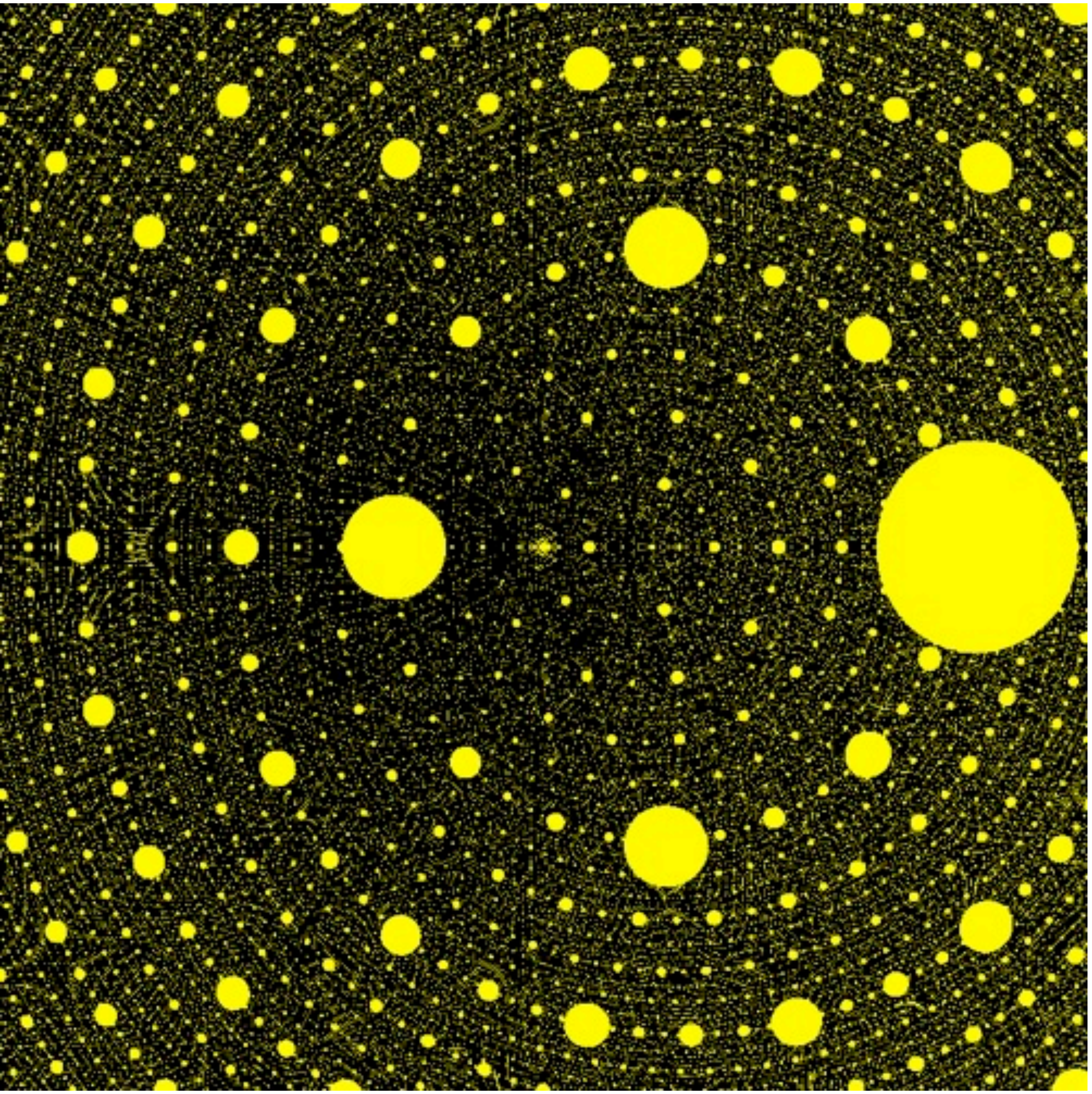
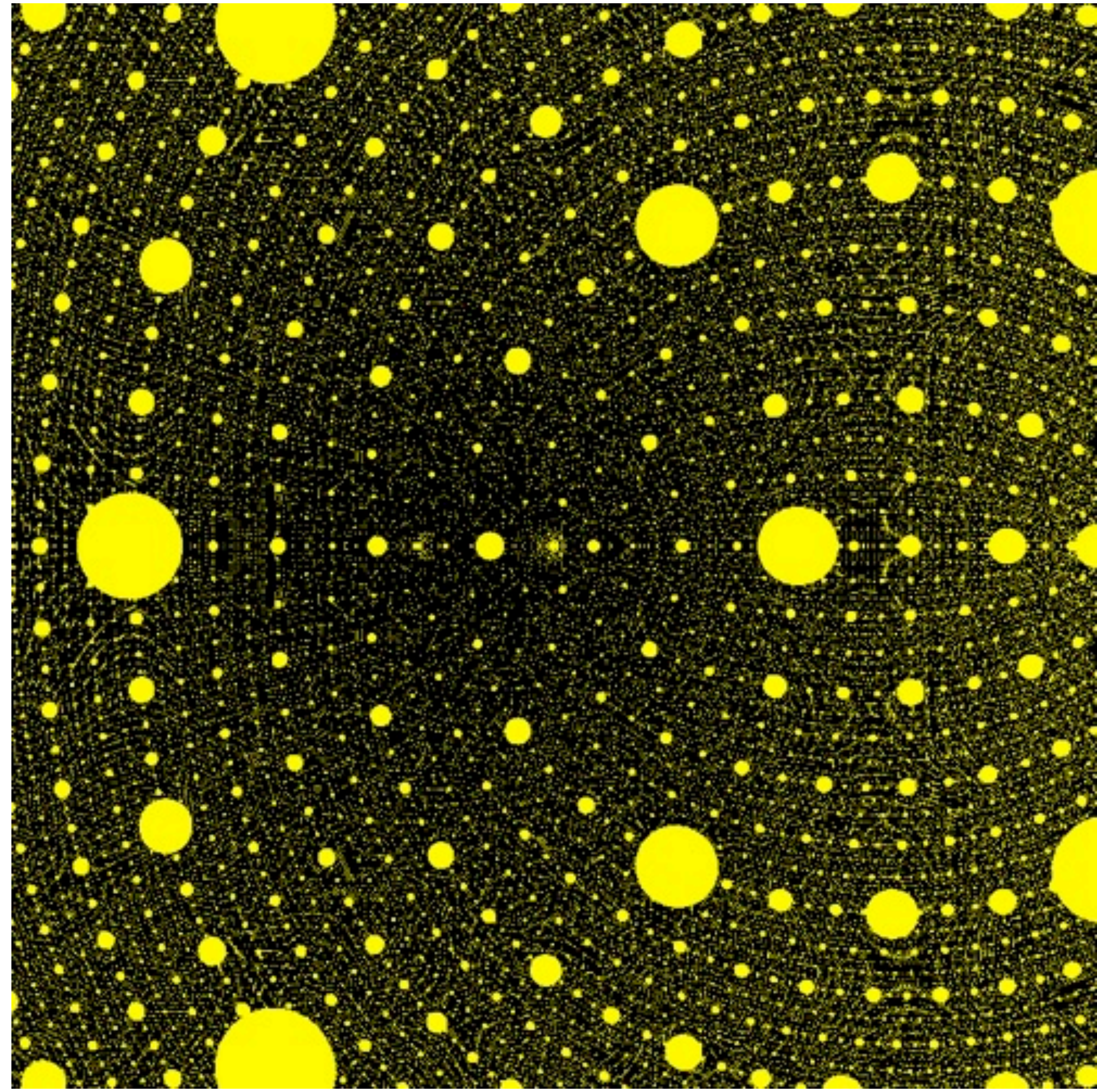



(2.0,0.0)

$a = 2$ $\qquad \mu_a = \mu_b$ if and only if $a = b$ $\qquad b = 5$





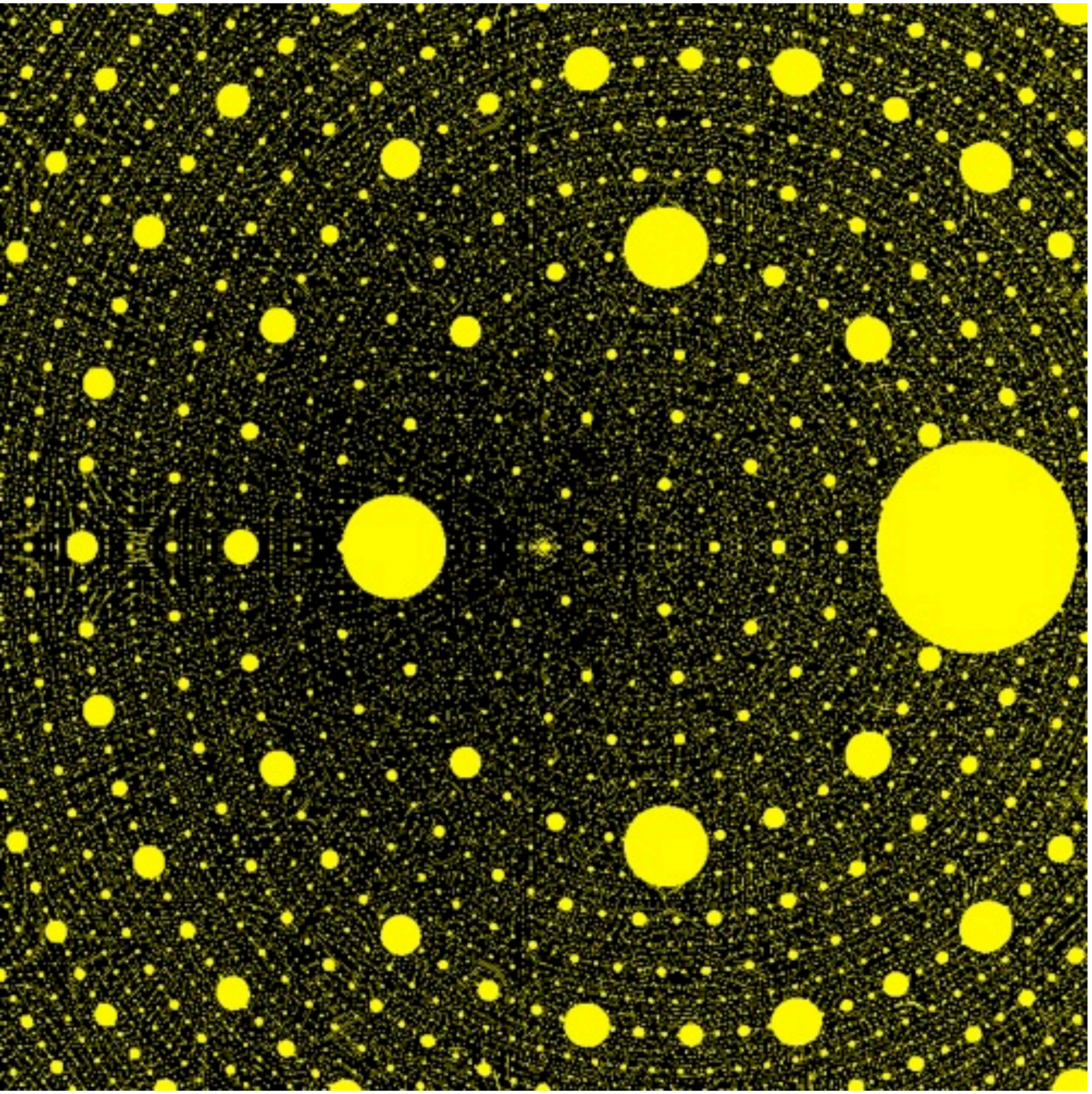
$$E_t = \{y^2 = x(x-1)(x-t)\}$$

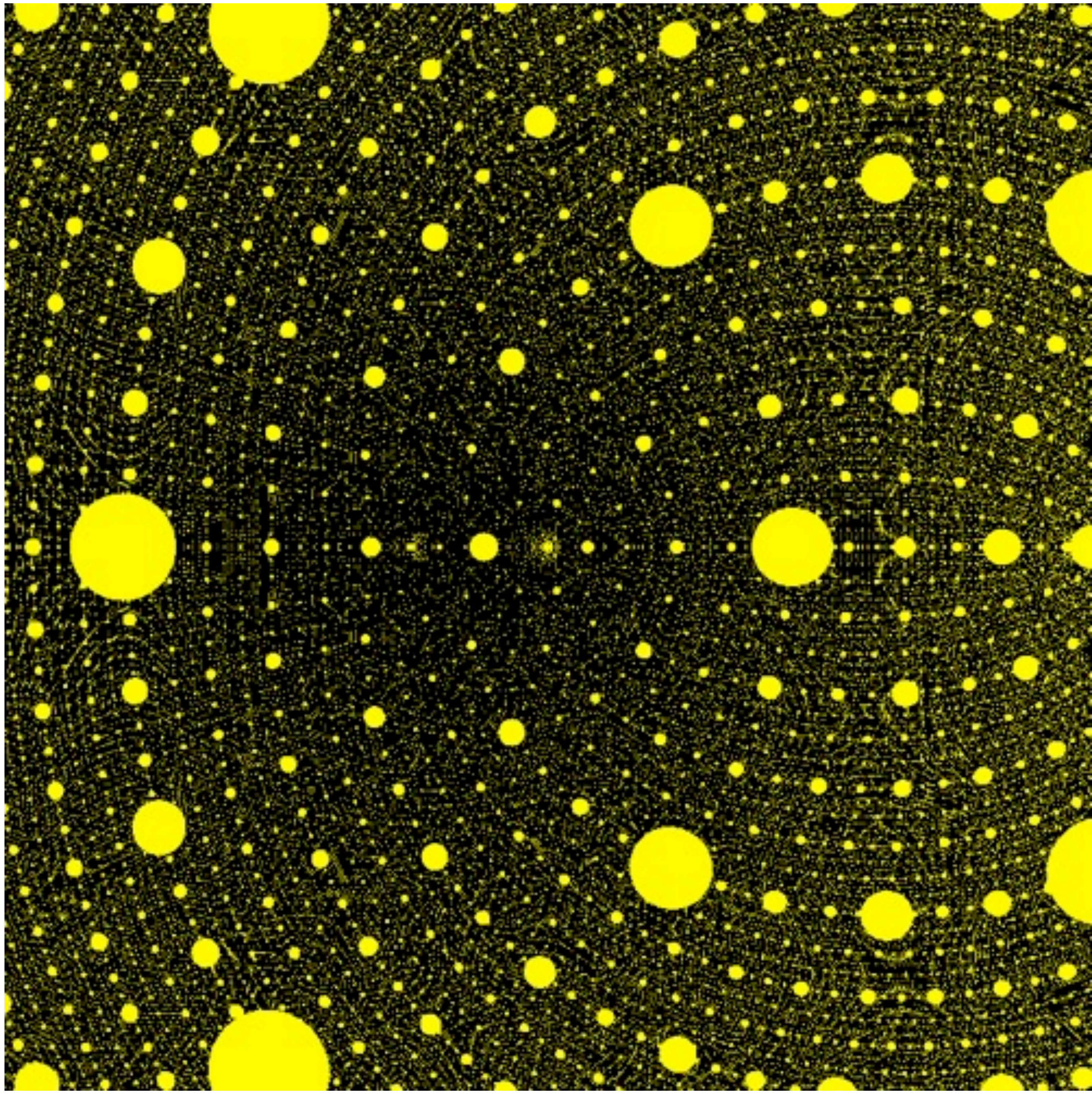
The Haar measure on E_t pushed down to $\mathbb{P}^1$ is

$$\mu_t = \frac{C(t)}{|z(z-1)(z-t)|}\,|dz|^2 \quad \text{where } C(t) = 2|t(t-1)|\rho_\Sigma(t).$$

Density for hyperbolic metric on triply-punctured sphere (McMullen)

$a = 2$ $\mu_a = \mu_b$ if and only if $a = b$ $b = 5$





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Potential function for μ_t: $\Longrightarrow$ Potential function for μ_a:

$$g_t(z) = 2C(t) \int_{\mathbb{P}^1} \frac{\log|z-\zeta|}{|\zeta(\zeta-1)(\zeta-t)|}\, |d\zeta|^2$$

$$g_a(t) = 2C(t) \int_{\mathbb{P}^1} \frac{\log|a-\zeta|}{|\zeta(\zeta-1)(\zeta-t)|}\, |d\zeta|^2$$

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Theorem. (Silverman) The components in the local decomposition

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for t near $t_0 \in X(\bar{K})$.

What do we know for dynamical canonical height?

Known for: Latt`es maps (a corollary of above)

Particular families of polynomials and rational maps

(Baker-D., D.-Wang-Ye, Ghioca-Hsia-Tucker, Ghioca-Mavraki, Ingram)

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False for general dynamical families! (D.-Wang-Ye, 2015)

for t near $t_0 \in X(\bar{K})$.

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A more basic question:
do we understand the leading terms?

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A more basic question:
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Fact. $\hat{h}_E(P)$ and $\hat{\lambda}_{E,t_0}(P)$ are rational numbers.

Explanation. These are intersection numbers on a Néron model.

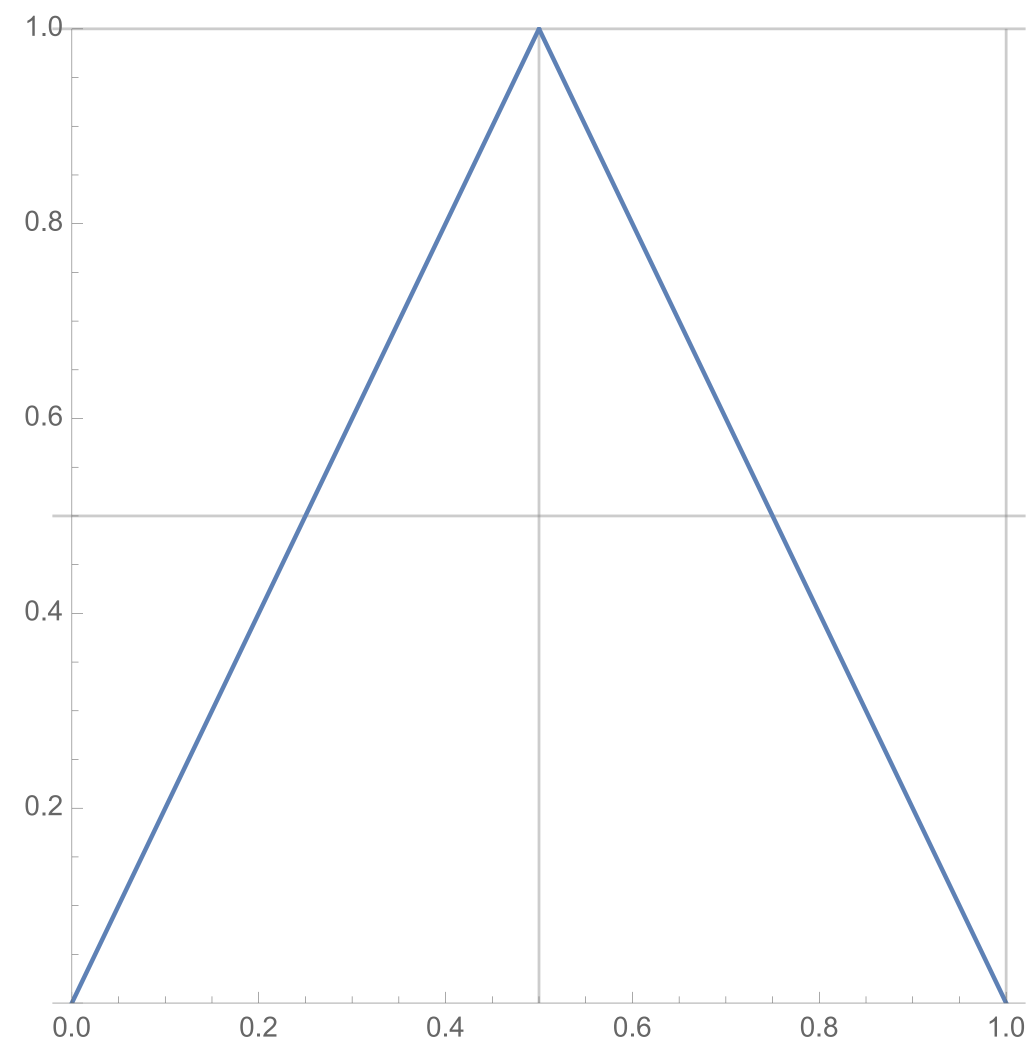
Another Fact. The analogous "weak" Néron models do not always exist in the dynamical setting. (Call-Silverman, Hsia)

Rationality of canonical height

(work in progress with Dragos Ghioca)

1. There is a dynamical proof that local/global canonical heights are always rational for elliptic curves.

Idea:
Dynamics of the multiplication-by-2 map on the Berkovich $\mathbf{P}^1$,
Julia set is an interval.
Action is by the tent map of slope 2,
all rational points are preperiodic.
(Favre, Rivera-Letelier)



The tent map of slope 2

Compare:

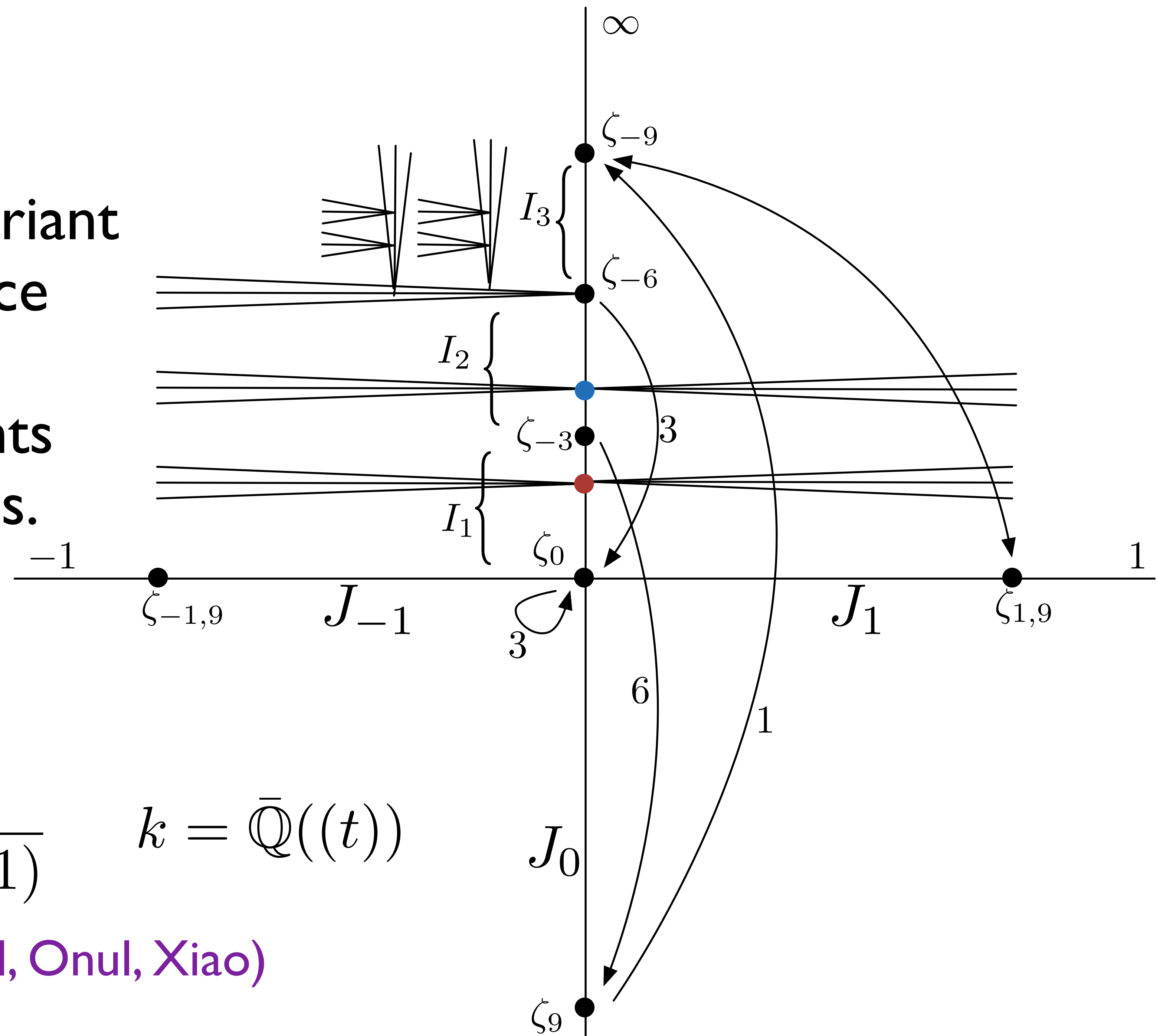
Theorem. (Ingram, 2012)
For polynomials, local heights at non-archimedean places are rational.

Rationality of canonical height

(work in progress with Dragos Ghioca)

2. There exist rational functions and points with irrational local heights!

Idea:
Julia set contains forward invariant intervals in the Berkovich space AND classical points.
There are Cantor sets of points containing aperiodic itineraries.



$$f_t(z) = \frac{t^{18}z^6+1}{t^{18}z^6+z(z-1)(z+1)} \qquad k = \bar{\mathbb{Q}}((t))$$

(Bajpai, Benedetto, Chen, Kim, Marschall, Onul, Xiao)

Rationality of canonical height

(work in progress with Dragos Ghioca)

2. There exist rational functions and points with irrational local heights! BUT, we expect these points to be transcendental...

(Fatou, Bell-Bruin-Coons, Adamczewski-Bell)

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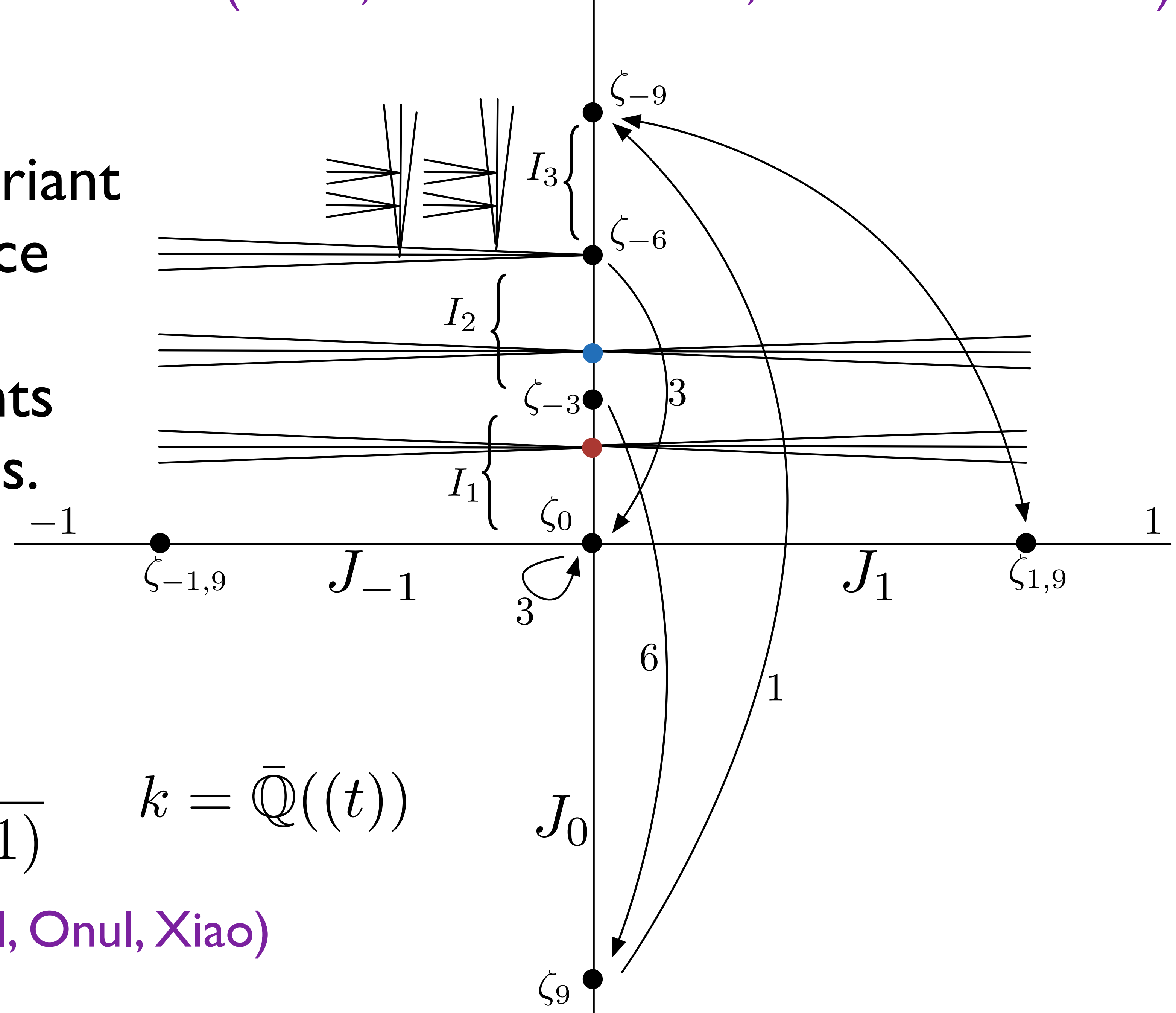

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Next steps

$$k = K(X),\ f : \mathbb{P}^1 \to \mathbb{P}^1 \text{ defined over } k,\ \ P \in \mathbb{P}^1(\bar{k})$$

Question. Is the canonical height $\hat{h}_f(P)$ rational?

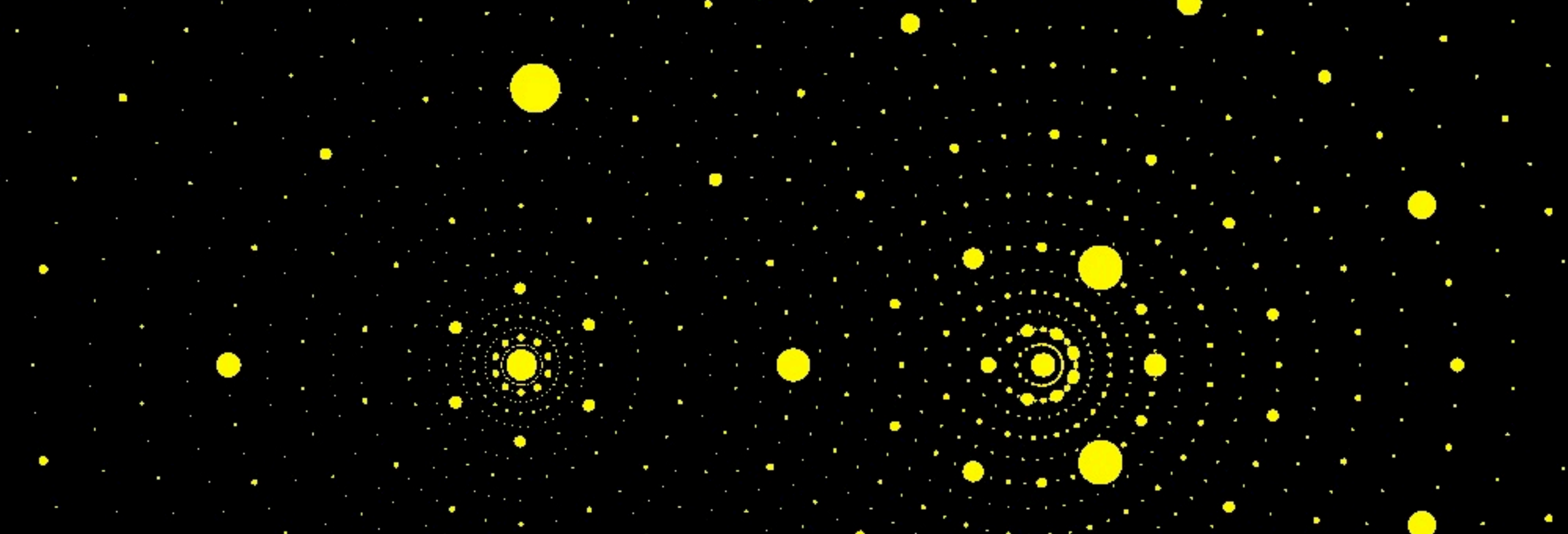
Question. Is there a good intersection-theoretic description of $\hat{h}_f(P)$, even in the absence of (weak) Néron models?

Question. Is there a divisor $D_P \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ so that

$$\hat{h}_{f_t}(P_t) = \ h_{X,D_P}(t)\ +\ O(1)$$

Question. Are the pieces in the local decomposition of $\hat{h}_{f_t}(P_t)$ "nice" functions of t?

Thank you, Joe, for providing so many great ideas and inspiration!



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