Determining Potential Good Reduction in Arithmetic Dynamics

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Notation

Throughout this talk, we set the following notation:

- K is a field
- $\overline{K}$ is an algebraic closure of K
- $|\cdot|$ is a non-archimedean absolute value on $\overline{K}$
- $\mathcal{O} = \{x \in K : |x| \leq 1\} \subseteq K$ is the ring of integers,
- $\mathcal{M} = \{x \in K : |x| < 1\} \subseteq K$ is the maximal ideal,
- $k = \mathcal{O}/\mathcal{M}$ is the residue field,
- $p = \operatorname{char} k \geq 0$ is the residue characteristic.

For example, $K = \mathbb{Q}_p$, $\overline{K} = \overline{\mathbb{Q}}_p$, $\mathcal{O} = \mathbb{Z}_p$, $k = \mathbb{F}_p$.

Or $K = \mathbb{F}((t))$, $\overline{K} = \overline{\mathbb{F}((t))}$, $\mathcal{O} = \mathbb{F}[[t]]$, $k = \mathbb{F}$.

Or $K = \overline{K} = \mathbb{C}_p$, $k = \overline{\mathbb{F}}_p$.

Dynamics on $\mathbb{P}^1(K)$

Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

[$\deg \phi := \max\{\deg f, \deg g\}$, where $\phi = f/g$ in lowest terms.]

Then $\phi : \mathbb{P}^1(\overline{K}) \to \mathbb{P}^1(\overline{K})$. Write $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}$.

Any linear fractional map $h(z) = \frac{az+b}{cz+d} \in \mathrm{PGL}(2, K)$ changes coordinates on $\mathbb{P}^1(\overline{K})$.

$$\begin{tikzcd}
{} \arrow[r, "\phi"] & \mathbb{P}^1 \arrow[r, "\phi"] \arrow[d, "h"] & \mathbb{P}^1 \arrow[r, "\phi"] \arrow[d, "h"] & \mathbb{P}^1 \arrow[r, "\phi"] \arrow[d, "h"] & \mathbb{P}^1 \arrow[r, "\phi"] \arrow[d, "h"] & {} \\
{} \arrow[r, "\psi"] & \mathbb{P}^1 \arrow[r, "\psi"] & \mathbb{P}^1 \arrow[r, "\psi"] & \mathbb{P}^1 \arrow[r, "\psi"] & \mathbb{P}^1 \arrow[r, "\psi"] & {}
\end{tikzcd}$$

The effect on ϕ is conjugation: $\psi = h \circ \phi \circ h^{-1}$.

Good Reduction

Given a polynomial $f \in \mathcal{O}[z]$, denote by $\overline{f}(z) \in k[z]$ the polynomial formed by reducing all coefficients of f modulo $\mathcal{M}$.

Definition (Morton, Silverman 1994)

Let $\phi(z) \in K(z)$ be a rational function. Write $\phi = f/g$ for $f, g \in \mathcal{O}[z]$ with at least one coefficient of f or g having absolute value 1.
Let $\overline{\phi} := \overline{f}/\overline{g}$. We say

- ϕ has *good reduction* (at v) if $\deg \overline{\phi} = \deg \phi$.
- ϕ has *bad reduction* (at v) if $\deg \overline{\phi} < \deg \phi$.
- ϕ has *potential good reduction* (at v) if there is some $h \in \mathrm{PGL}(2, \overline{K})$ such that $h \circ \phi \circ h^{-1} \in \overline{K}(z)$ has good reduction.

Reduction Examples

Example. $\phi(z) = \dfrac{z^3 - 2z}{z^4 + 1} \in \mathbb{Q}_5(z)$ has $\deg \phi = 4$.

But $\overline{\phi}(z) = \dfrac{z(z^2 - 2)}{(z^2 + 2)(z^2 - 2)} = \dfrac{z}{z^2 + 2} \in \mathbb{F}_5(z)$ has $\deg \overline{\phi} = 2 < 4$, so ϕ has bad reduction.

Example. $\phi(z) = z^2 - \dfrac{7}{144} = \dfrac{144z^2 - 7}{144}$
has bad reduction at $p = 2$ and $p = 3$,
since $\overline{\phi}(z) = \dfrac{-7}{0}$ has degree $0 < 2$.

But ϕ has good reduction at $p = 5, 7, 11, \ldots$.

Potential Good Reduction Examples

Example. $\phi(z) = pz^2 \in \mathbb{Q}_p(z)$ has bad reduction: $\overline{\phi}(z) = 0$.

But $\psi(z) := p\phi\left(\dfrac{z}{p}\right) = z^2 \in \mathbb{Q}_p(z)$ has good reduction.

Example. $\phi(z) = pz^3 \in \mathbb{Q}_p(z)$ has bad reduction: $\overline{\phi}(z) = 0$.

But $\psi(z) := \sqrt{p}\phi\left(\dfrac{z}{\sqrt{p}}\right) = z^3 \in \overline{\mathbb{Q}}_p(z)$ has good reduction.

Example. $\phi(z) = z^2 - \dfrac{1}{2} \in \mathbb{Q}_2(z)$ has bad reduction: $\overline{\phi}(z) = 1/0$.

But $\psi(z) := \phi\left(z + \dfrac{1+i}{2}\right) - \dfrac{1+i}{2} = z^2 + (1+i)z - 1 \in \overline{\mathbb{Q}}_2(z)$ has good reduction.

Periodic points and multipliers

Definition

Let $x \in \mathbb{P}^1(\overline{K})$ such that $\phi^n(x) = x$ for some (minimal) $n \geq 1$. Then we say x is *periodic* of (exact) period n.

The *multiplier* of x is $\lambda := (\phi^n)'(x) \in \overline{K}$. We say x is

- *repelling* if $|\lambda| > 1$,
- *attracting* if $|\lambda| < 1$, or
- *indifferent* if $|\lambda| = 1$.

Theorem (Morton, Silverman, 1995)

If ϕ has potential good reduction, then all periodic points of ϕ are attracting or indifferent.

But Not Conversely

However, there **are** maps with no repelling periodic points but also **not** potentially good.

Example $K = \mathbb{Q}_p$, $\phi(z) = z^{2p^2} + \frac{1}{p} z^{p^2}$ has this property.

Example Let E be an elliptic curve of multiplicative reduction. Then $[2] : E \to E$ induces a *Lattès map* $\phi : \mathbb{P}^1(\overline{K}) \to \mathbb{P}^1(\overline{K})$ with $x([2]P) = \phi(x[P])$.

See page 59 of *The Arithmetic of Elliptic Curves*.

ϕ is a quartic rational function with no repelling periodic points but that is not potentially good.

E.g. $\quad y^2 + xy = x^3 + p \quad$ gives $\quad \phi(z) = \dfrac{z^4 - 8pz - p}{4z^3 + z^2 + 4p}$.

Good Reduction and disks

To say that $\phi(z) \in K(z)$ has good reduction is to say that $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ extends to a $\operatorname{Spec}\mathcal{O}$-morphism from $\mathbb{P}^1_{\mathcal{O}}$ to itself.

To explain: writing $D(a,r) := \{x \in \overline{K} : |x-a| < r\}$, we can partition $\mathbb{P}^1(\overline{K})$ into residue classes: i.e., the open unit disks

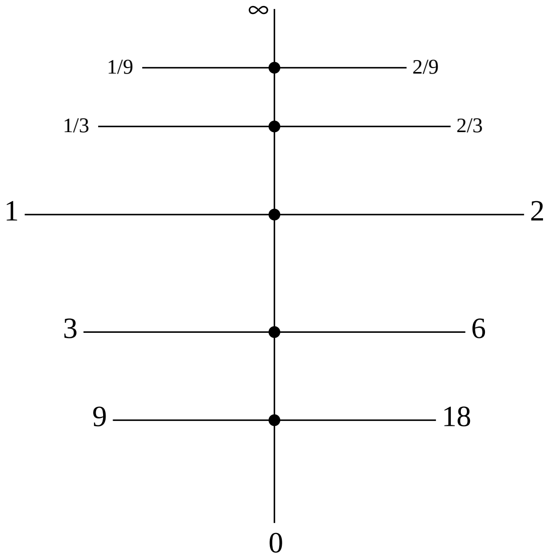
$$D(a,1) \quad \text{for } |a| \leq 1$$

and the "disk at infinity" $D(\infty,1) := \{x \in \mathbb{P}^1(\overline{K}) : |x| > 1\}$.

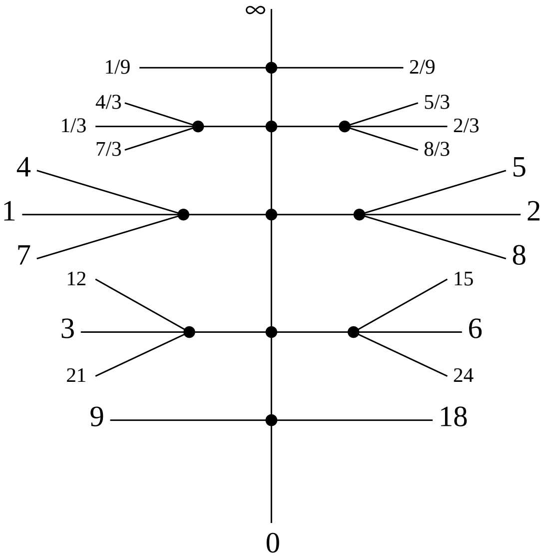
(Reduction-mod-$\mathcal{M}$ maps $\mathbb{P}^1(\overline{K}) \to \mathbb{P}^1(\overline{k})$ by $D(a,1) \to \overline{a}$.)

To say that ϕ has good reduction is to say that ϕ maps every residue class $D(a,1)$ into (and in fact, onto) the residue class $D(\phi(a),1)$.

Building the Berkovich Projective Line



Building the Berkovich Projective Line



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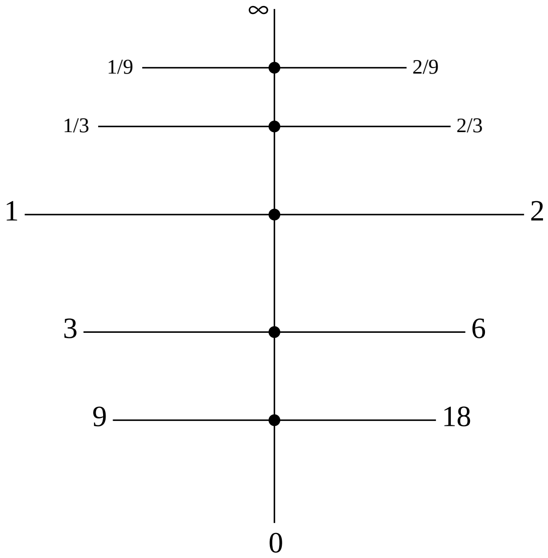

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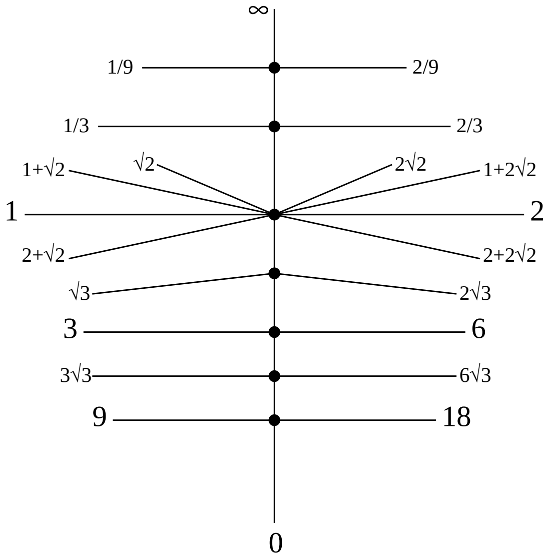
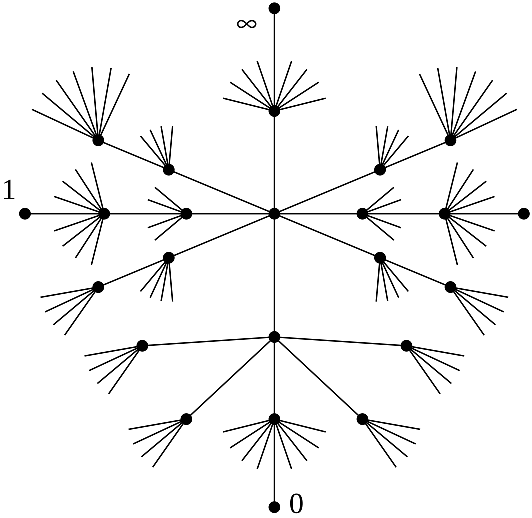


The Berkovich Projective Line $\mathbb{P}^1_{\mathrm{Ber}}$



Good Reduction and the Berkovich Projective line

$\phi \in K(z)$ extends to a (continuous) map $\phi : \mathbb{P}^1_{\mathrm{Ber}} \to \mathbb{P}^1_{\mathrm{Ber}}$.
The point $\zeta(0,1)$ corresponding to the closed unit disk is called the **Gauss point**.

Theorem (Rivera-Letelier)

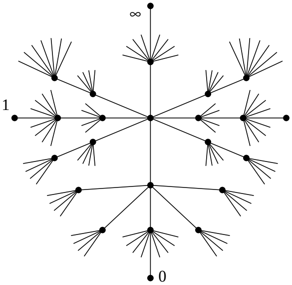
ϕ has good reduction if and only if the Gauss point is a **totally invariant fixed point**, *i.e., $\phi^{-1}\big(\zeta(0,1)\big) = \{\zeta(0,1)\}$.*

Corollary

ϕ has potential good reduction if and only if there is some $a \in \overline{K}$ and $r \in |\overline{K}^{\times}|$ such that $\zeta(a,r)$ is a totally invariant fixed point.

Changing Coordinates

Any $\zeta(a, r) \in \mathbb{P}^1_{\mathrm{Ber}}$ with $r \in |\overline{K}^\times|$ can be moved to $\zeta(0, 1)$ by a coordinate change in $\mathrm{PGL}(2, \overline{K})$.



Pick x_1, x_2, x_3 in separate branches emanating from $\zeta(a, r)$.
Pick $h \in \mathrm{PGL}(2, \overline{K})$ with $h(x_1) = 0$, $h(x_2) = 1$, $h(x_3) = \infty$.
Then $h(\zeta(a, r)) = \zeta(0, 1)$.

Rumely, 2013: gives an algorithm for determining whether or not ϕ has potential good reduction.

A Lemma on Local Dynamics

Lemma

If ϕ has good reduction, and if the residue class $D(a,1)$ is fixed by ϕ, then one of the following is true:

- $D(a,1)$ *is an* **attracting component***:*
  it contains an attracting fixed point b,
  and $\lim_{n\to\infty} \phi^n(x) = b$ *for all* $x \in D(a,1)$.
- $D(a,1)$ *is an* **indifferent component***:*
  $\phi : D(a,1) \to D(a,1)$ *is one-to-one*

So (for good reduction):

- an attracting fixed point can't share its residue class with another fixed point.
- an indifferent fixed point can't share its residue class with one of its preimages.

A Fixed-Point Criterion for Potential Good Reduction

Theorem (RB, 2014)

Let $\phi \in K(z)$ *with* $d := \deg \phi \geq 2$.
Let $x_1, \ldots, x_{d+1} \in \mathbb{P}^1(\overline{K})$ *be the fixed points of* ϕ.

- *If any* x_i *is repelling, then* ϕ *is not potentially good.*
- *If* x_1 *is indifferent, we can choose* $y_1 \in \phi^{-1}(x_1)$ *and* $y_2 \in \phi^{-1}(y_1)$ *with* x_1, y_1, y_2 *all distinct.*
  Let $h \in \mathrm{PGL}(2, \overline{K})$ *with* $h(x_1) = 0$, $h(y_1) = 1$, *and* $h(y_2) = \infty$.
  Then ϕ *is potentially good if and only if* $h \circ \phi \circ h^{-1}$ *has good reduction.*
- *If* x_1 *and* x_2 *are attracting, then* x_1, x_2, x_3 *are all distinct, so there is a unique* $h \in \mathrm{PGL}(2, \overline{K})$ *with* $h(x_1) = 0$, $h(x_2) = 1$, *and* $h(x_3) = \infty$.
  Then ϕ *is potentially good if and only if* $h \circ \phi \circ h^{-1}$ *has good reduction.*

How Big an Extension Do We Need?

If ϕ *does* have potential good reduction, the minimal field of definition L of the coordinate change $h \in \mathrm{PGL}(2, \overline{K})$ chosen in the theorem could have $[L : K] = d^3 - d$, *a priori*.

Theorem (RB, 2014)

Let $p = \operatorname{char} k \geq 0$ *be the residue characteristic of* K*. Let* $\phi \in K(z)$ *with* $\deg \phi = d \geq 2$*. Let*

$$B := \max\{\, p^{v_p(d)}(d-1),\ p^{v_p(d-1)}d,\ d+1 \,\}.$$

If ϕ *has potential good reduction, then* ϕ *attains good reduction over some field* L *with* $[L : K] \leq B$*.*

A Key Case

Suppose that ϕ has potential good reduction,
with totally invariant Berkovich point ξ,
but that all points of $\mathbb{P}^1(K)$ lie in the same branch U from ξ.

Then this branch/disk U is a fixed component, and hence either attracting or indifferent. **Let's assume it's attracting**.

Then exactly d of the $d+1$ fixed points lie outside U.

If K is complete, the monic polynomial f with those d points as its roots has coefficients in K.

Even if K is not complete, we can show there is a polynomial $f \in K[z]$ with $\deg f = d$ and all roots of f outside U.

A Key Case, Continued

Let $q := p^{v_p(d)}$. [Or $q := 1$ if $p = 0$.] A further argument shows there is a polynomial $g \in K[z]$ with $\deg g = q$ and all roots of g outside U.

Let $\alpha \in \overline{K}$ be a root of g, so that $[K(\alpha) : K] \leq q$.

Note: there are $K(\alpha)$-rational points in at least two different branches from ξ.

So some $K(\alpha)$-rational coordinate change moves U to the open disk $D(0, r)$ for some $r > 0$.

Using the fact that U is attracting, we can show that $r \in |K(\alpha)^\times|^{1/n}$ for some $n \leq d - 1$.
So there's a field L with $[L : K(\alpha)] = n$ for which three different branches off ξ contain L-rational points.

One Other Case

Suppose that ϕ has potential good reduction,
with totally invariant Berkovich point ξ,
and $\mathbb{P}^1(K)$ intersects exactly two branches U and V from ξ,
and $\phi(U) = V$ and $\phi(V) = U$.

Let $f \in K[z]$ be the degree-$(d+1)$ polynomial whose roots are the fixed points of ϕ, and let α be a root of f.

Note that $\alpha \notin U \cup V$.

So $[K(\alpha) : K] \leq d + 1$, and three different branches off ξ contain $K(\alpha)$-rational points.

(Recall $B := \max\{\ p^{v_p(d)}(d-1),\ p^{v_p(d-1)}d,\ d+1\ \}$.)

The Bound is Sharp: The $p|d$ Case

Assume K is discretely valued with uniformizer π.

Example. $d = mp^e$ with $p \nmid m$. Let $q := p^e$, and let

$$\phi(z) := z + \left(\pi^{-1} z^q + 1\right)^m \in K[z].$$

Then $\deg \phi = d$, and the Berkovich point $\xi := \zeta\left(\pi^{1/q}, |\pi|^{m/(d-1)}\right)$ is totally invariant.

Any coordinate change moving ξ to $\zeta(0,1)$ requires an extension L/K with ramification degree at least $q(d-1)$.

Note, of course, the attracting fixed point at ∞.

The other $d-1$ fixed points are indifferent, and they lie in the disk $\overline{D}\left(\pi^{1/q}, |\pi|^{m/(d-1)}\right)$ corresponding to ξ.

The Bound is Sharp: The $p|(d-1)$ Case

Assume K is discretely valued with uniformizer π.

Example. $d-1 = mp^e$ with $p \nmid m$. Let $q := p^e$, and let

$$\phi(z) := z + \frac{\pi^{d-1}}{(z^q - \pi^{q-1})^m} \in K(z).$$

Then $\deg \phi = d$, and the Berkovich point
$\xi := \zeta\big(\pi^{(q-1)/q}, |\pi|^{(d-1)/d}\big)$ is totally invariant.

Any coordinate change moving ξ to $\zeta(0,1)$ requires an extension L/K with ramification degree at least qd.

- The branch U containing $\mathbb{P}^1(K)$ is indifferent and contains **all** of the fixed points (which are all at ∞).
- Another **single** branch $V = D(\pi^{(q-1)/q}, |\pi|^{(d-1)/d})$ contains all the other $d-1$ preimages of the points in U.
- So we need to take preimages of points in V — thus a further degree-d extension — to realize a third branch.

The Bound is Sharp: The $d+1$ Case

Assume K is discretely valued with uniformizer π.

Example. Let

$$\phi(z) := \frac{\pi}{z^d}.$$

Then $\deg \phi = d$, and the Berkovich point $\xi := \zeta(0, |\pi^{1/(d+1)}|)$ is totally invariant.

Any coordinate change moving ξ to $\zeta(0,1)$ requires an extension L/K with ramification degree at least $d+1$.

- The disk $U := D(0, |\pi^{1/(d+1)}|)$ maps d-to-1 onto V, the complement of $\overline{D}(0, |\pi^{1/(d+1)}|)$.
- And V maps d-to-1 onto U. So we have a totally invariant attracting 2-cycle of components.
- To pick up another branch, we need to adjoin a fixed point, requiring degree $d+1$.