Interpreting Euclidean structure in the ample cone for elliptic K3 surfaces

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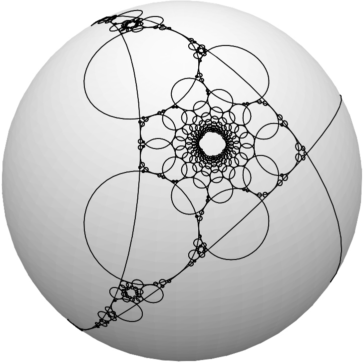
University of Nevada Las Vegas

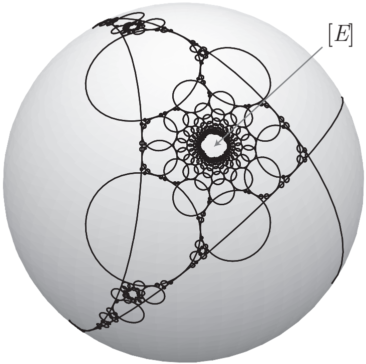
Happy Birthday Joe!
Silvermania, August 13th, 2015

The ample cone



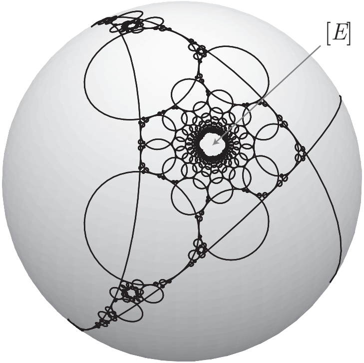
A hyperbolic cross section of an ample cone for a particular K3 surface X with an elliptic fibration with a section.

The ample cone



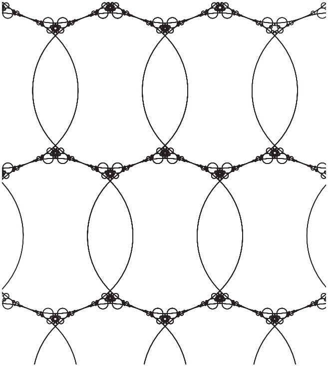
The accumulation point $[E]$ represents a divisor class in $\mathrm{Pic}(X)$ with $[E] \cdot [E] = 0$, an elliptic fibration of X.

The ample cone



Let us unfold this Poincaré ball into the Poincaré upper half space, with $[E]$ the point at infinity ...

The ample cone



… and we get this, which has a visible Euclidean structure.

The Neron-Tate pairing

- Recall the canonical height on an elliptic curve E:

$$\hat{h}(P) = \lim_{n\to\infty} \frac{h([n]P)}{n^2}.$$

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$$\hat{h}(P) = \lim_{n \to \infty} \frac{h([n]P)}{n^2}.$$

- And the Neron-Tate pairing:

$$\langle P, Q \rangle = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q),$$

which is a symmetric positive definite bilinear form.

Questions

Question 1

Can we explain the Euclidean structure of the ample cone?

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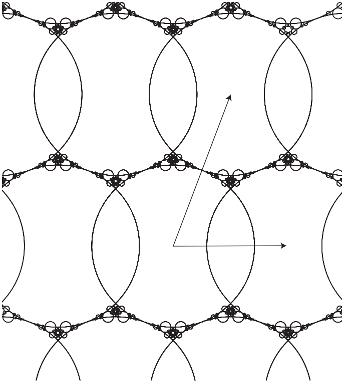
Question 2

Is there a natural underlying geometry to the Neron-Tate pairing?

Result



The translational symmetry corresponds to the additive group on the elliptic curves in the fibration.

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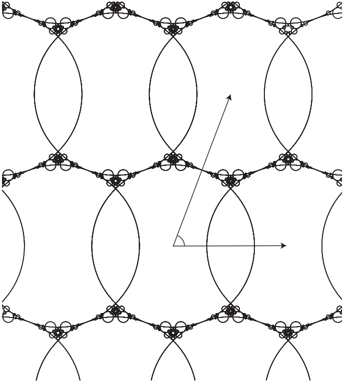

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The relative geometry comes from the Neron-Tate pairing.

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- There is a light cone:

$$\mathcal{L} = \{\vec{x} \in \mathrm{Pic}(X) \otimes \mathbb{R} : \vec{x} \cdot \vec{x} > 0, D \cdot \vec{x} > 0\},$$

where D is an ample divisor.

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$$\mathcal{K} = \{\vec{x} \in \mathcal{L} : \vec{x} \cdot C > 0 \text{ for all } C \in \mathcal{E}_{-2}.\}$$

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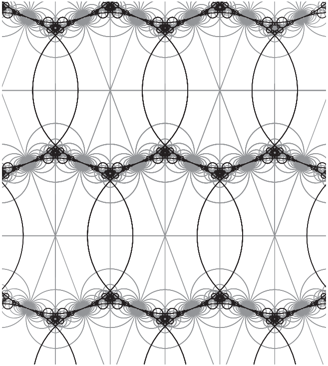
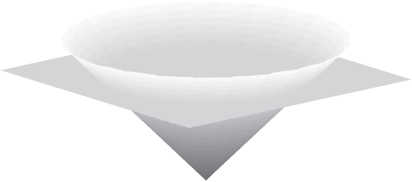
- Γ is large, in the sense that there exists a finite set S such that $\mathcal{E}_{-2} = \Gamma S$.



Symmetries of the ample cone.

Cross sections: An example with $\rho = 3$



A planar cross section ...

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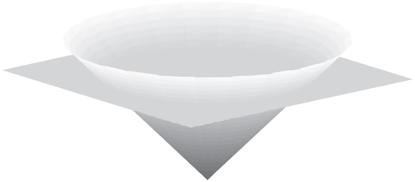

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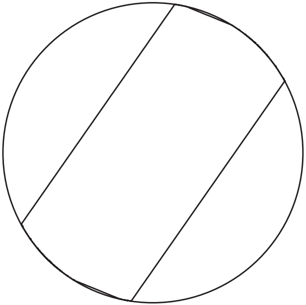
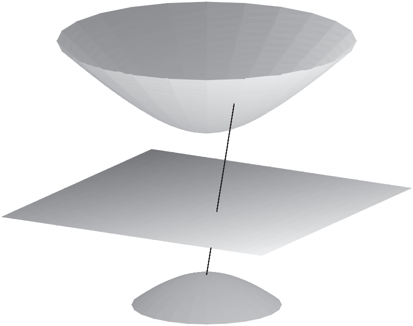


... gives this.

Projecting onto the Poincaré disc



- Intersect with a hyperboloid.

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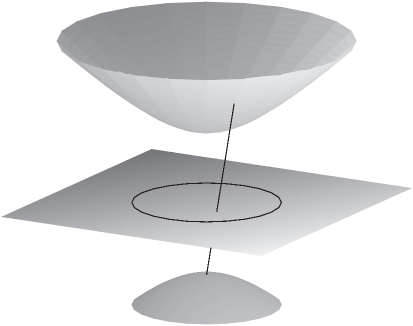
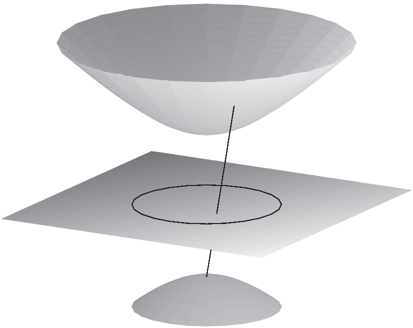
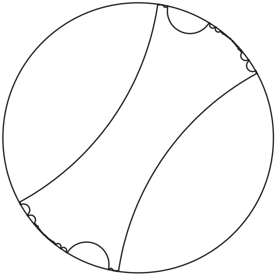


- Intersect with a hyperboloid.
- Project onto the unit disc.

Projecting onto the Poincaré disc





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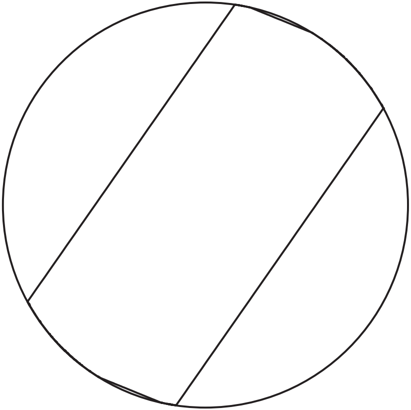
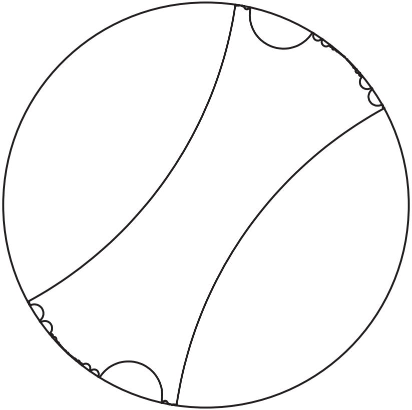
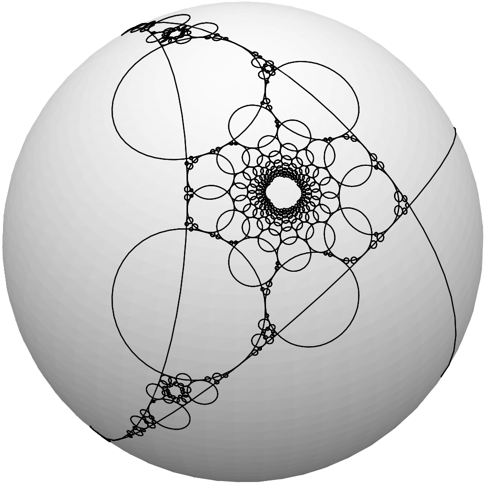


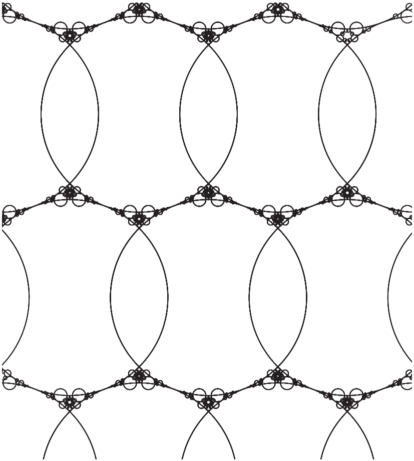
The Beltrami-Klein model



The Poincaré model

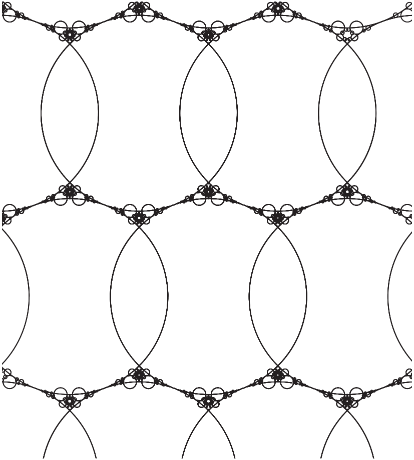


A hyperbolic cross section for an example with $\rho = 4$.



Unfolded

- Euclidean structure



Unfolded

- Euclidean structure
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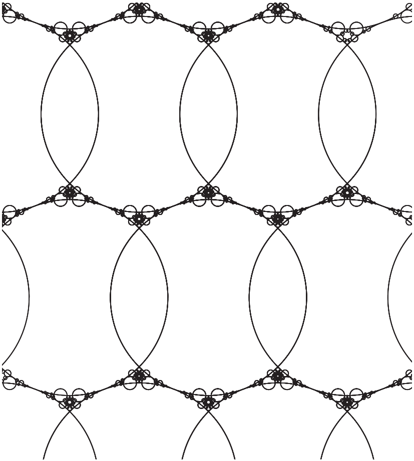

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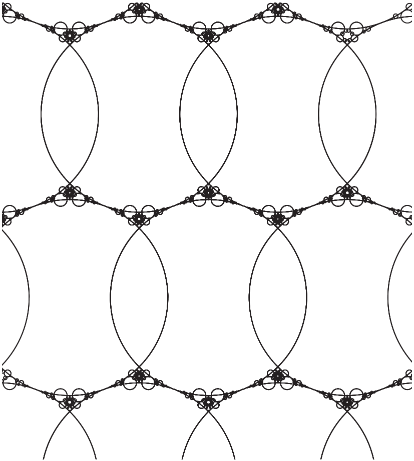
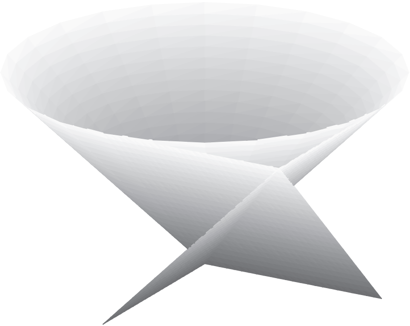
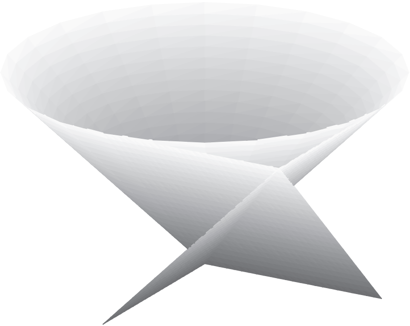
  - $\partial\mathbb{H}^m \setminus \{P\}$ has a Euclidean structure.



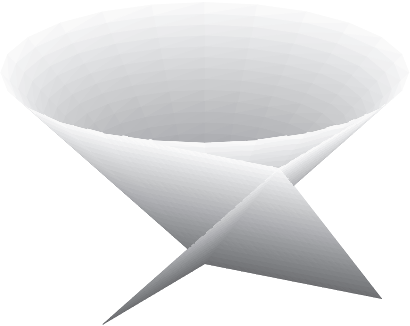
Unfolded

- Euclidean structure
- This is true in general:
  - $\partial\mathbb{H}^m \setminus \{P\}$ has a Euclidean structure.
  - $\mathcal{O}_P$, the stabilizer of P, acts as a group of Euclidean isometries.





- The light cone modulo rays through the origin is $\partial\mathbb{H}^m$.



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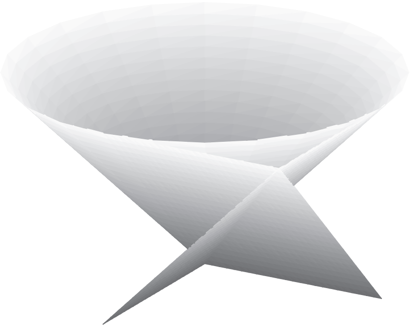
- Take two rays on the cone, representing two points at infinity, and consider their tangent planes.



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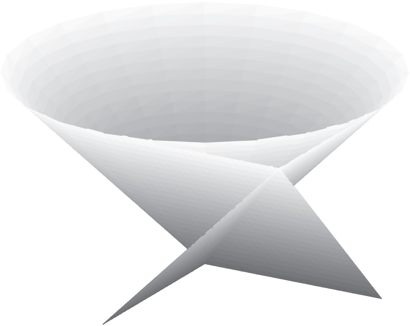
- Where the tangent planes intersect is a subspace where the intersection pairing is negative-definite.



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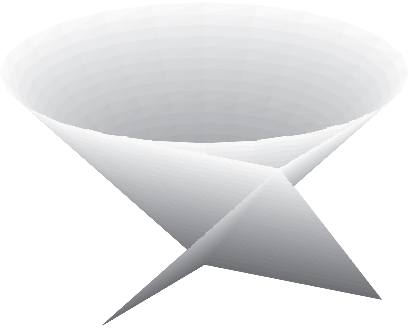

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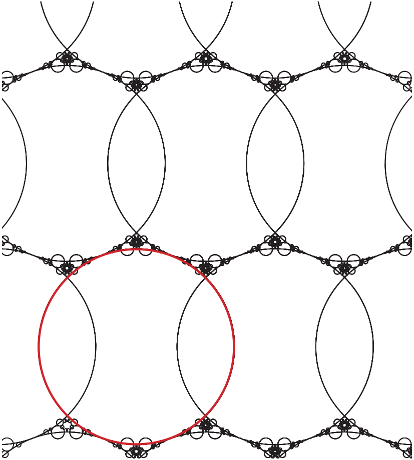
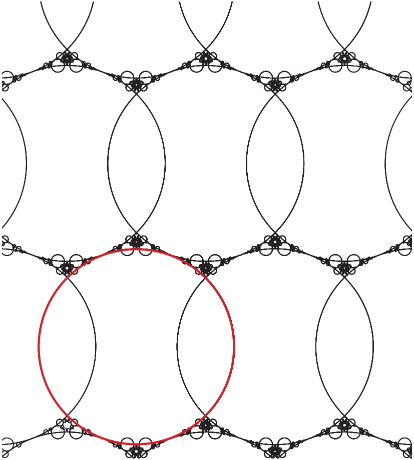
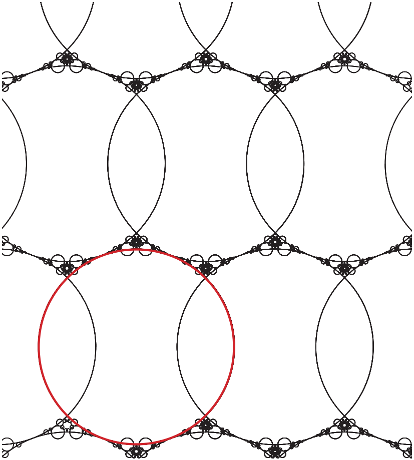
- Hence yields an inner product and a Euclidean structure.
- The rays represent the origin and the point at infinity.





- This plane (red circle), $D_1 \cdot \vec{x} = 0$, represents a section. I.e. $[E] \cdot D_1 = 1$.



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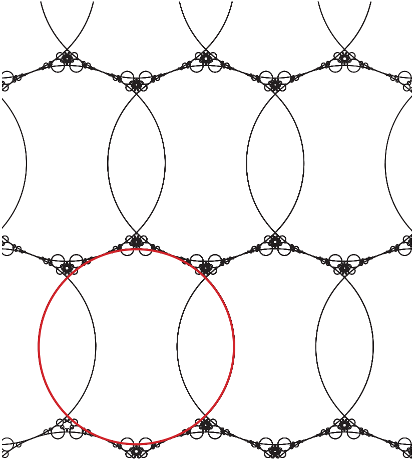

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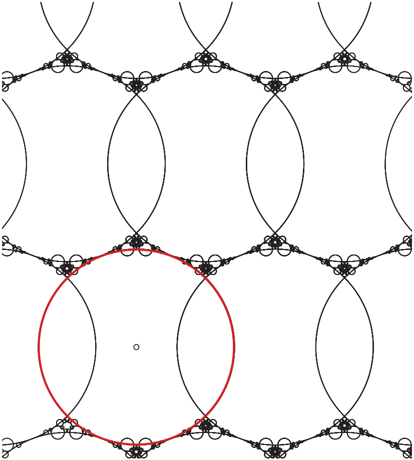
$$\sigma(P) = -P.$$



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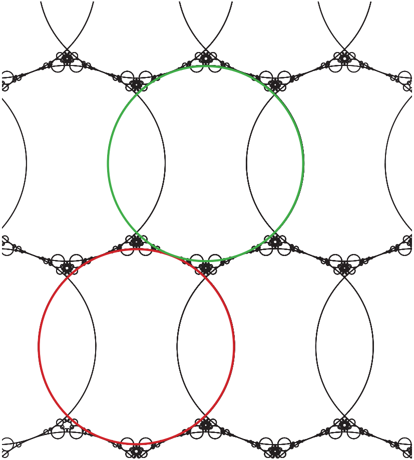
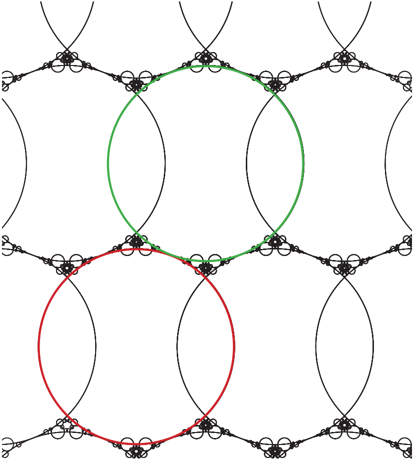
- σ^* is the -1 map on the Euclidean space, with origin $D_1 + [E]$, the center of the red circle.





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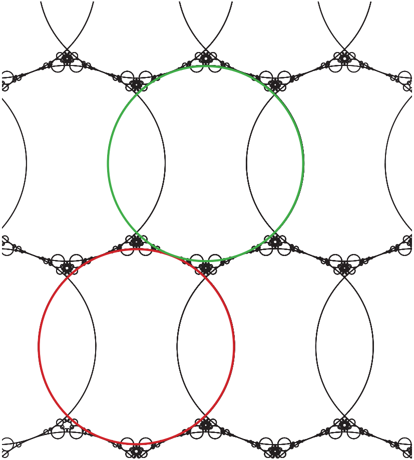

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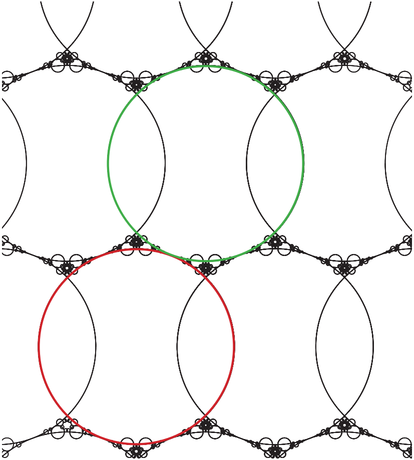

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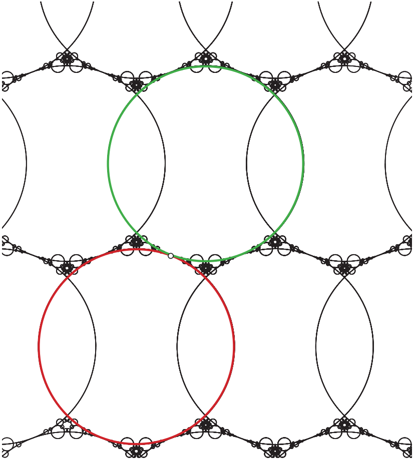

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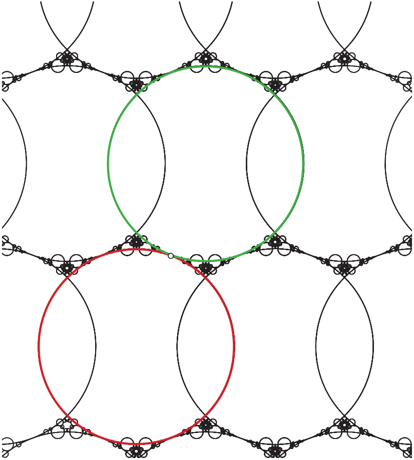

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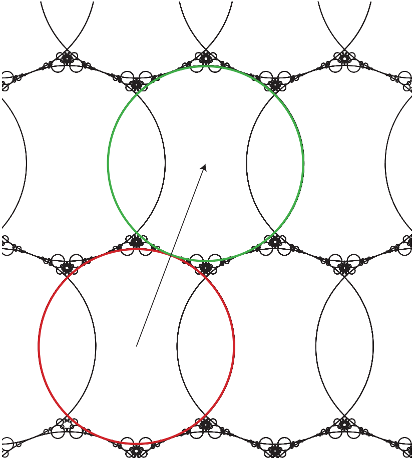

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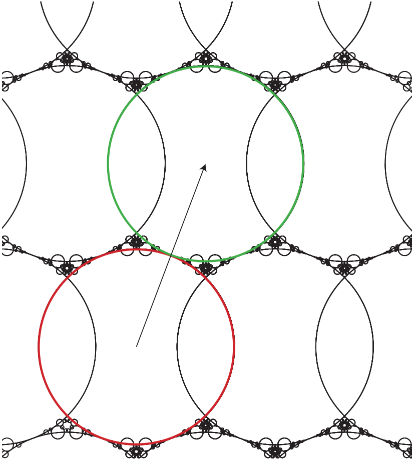

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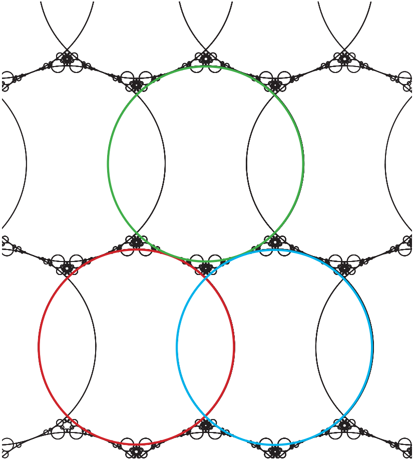
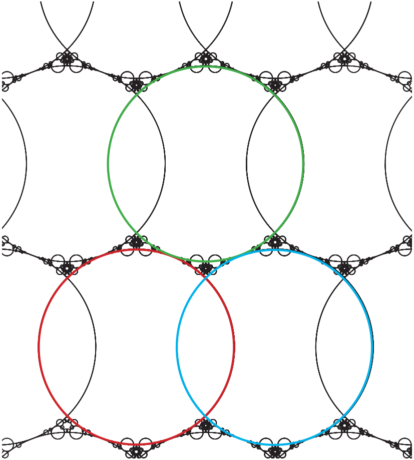
- $\tau_1^*(\vec{x}) = \vec{x} + (\vec{x}\cdot\vec{v}_1)[E] + \frac{1}{2}(\vec{x}\cdot[E])(\vec{v}_1\cdot\vec{v}_1)[E] + (\vec{x}\cdot[E])\vec{v}_1$.





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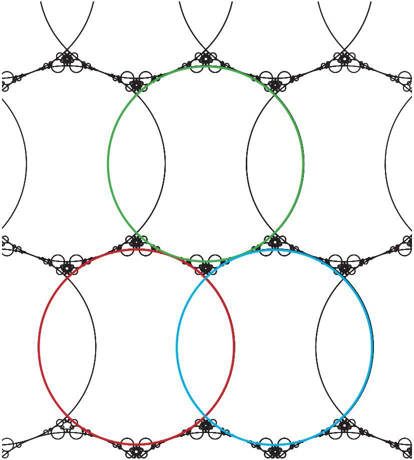

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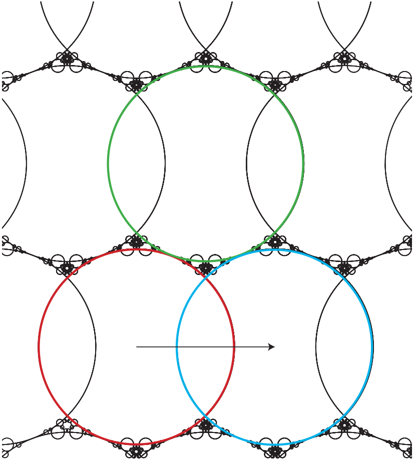

- And another section.
- $\sigma_2(P) = Q_2 - P$.
- $\tau_2^* = (\sigma_2\sigma)^*$ is another translation (red $\mapsto$ blue).

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- $\mathcal{D} = \{D_1, ..., D_\rho\}$ a basis of $\mathrm{Pic}(X)$.
- $\mathcal{D}^* = \{D_1^*, ..., D_\rho^*\}$ the dual basis, where $D_i \cdot D_j^* = \delta_{ij}$.
- $$\vec{h} : X \to \mathrm{Pic}(X) \otimes \mathbb{R}$$ $$\vec{h}(P) = \sum_{i=1}^{\rho} h_{D_i}(P) D_i^*.$$
- Nice properties:
  - $h_D(P) = \vec{h}(P) \cdot D + O(1)$.
  - $\vec{h}(\sigma P) = \sigma_* \vec{h}(P) + \vec{O}(1)$.

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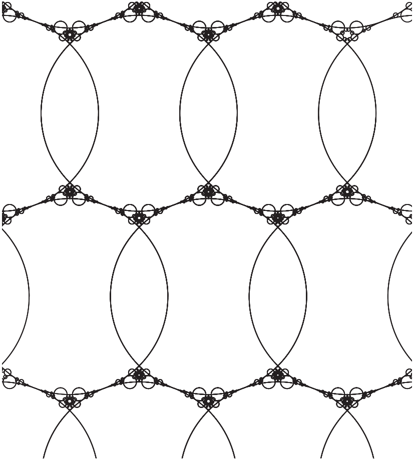
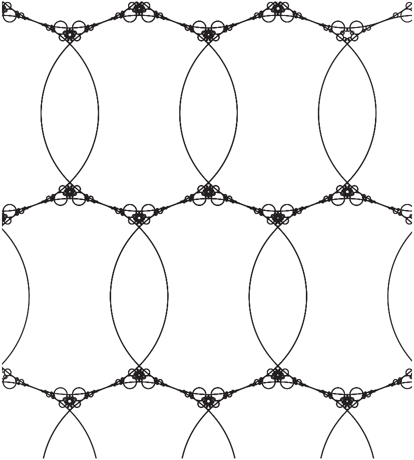
- Thus, as E grows in height in the divisor class $[E]$, the Neron-Tate pairing approaches the intersection pairing.





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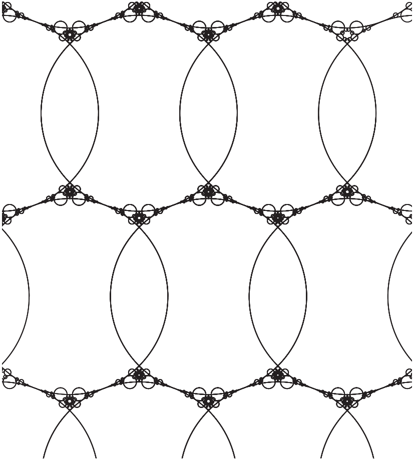

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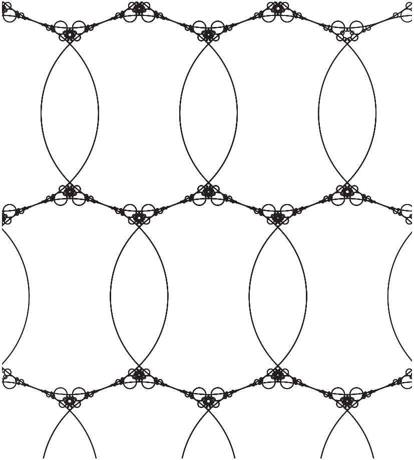
  - and such that $\Gamma_{[E]}$ has a subgroup isomorphic to $\mathbb{Z}^{\rho-2}$.



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Thank you.