CSPs in the Descriptive Context

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I. Descriptive combinatorics

Let G be a graph on the vertex set $V(G)$.

Let $n \in \{1, 2, \ldots, \aleph_0\}$. An *n-coloring* of G is a map $c : V(G) \to n$ with

$$\forall x, y \in V(G)\ xGy \implies c(x) \neq c(y).$$

The *chromatic number* of G is the minimal n for which G has an n-coloring. Notation: $\chi(G)$.

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Here, n is equipped with the discrete Borel structure.

The **Borel** *chromatic number* of G is the minimal n for which G has an n-coloring. Notation: $\chi_B(G)$.

Example: Rotations

$V(G) = S^1$. Let R_α be a rotation by an irrational angle α and

$$xGy \iff x = R_\alpha(y) \text{ or } y = R_\alpha(x).$$

Claim

$2 = \chi(G) < \chi_B(G) = 3.$

Example: Shift

Let $[\mathbb{N}]^{\mathbb{N}}$ denote the set of infinite subsets of $\mathbb{N}$.

Definition

Define the map $\mathcal{S}$ by

$$\mathcal{S}(x) = x \setminus \{min\ x\},$$

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Theorem (Galvin-Prikry)

For any $k \in \mathbb{N}$ *and any Borel coloring* $c : [\mathbb{N}]^{\mathbb{N}} \to k$ *there exits an* $A \in [\mathbb{N}]^{\mathbb{N}}$ *such that* c *is constant on* $[A]^{\mathbb{N}}$*.*

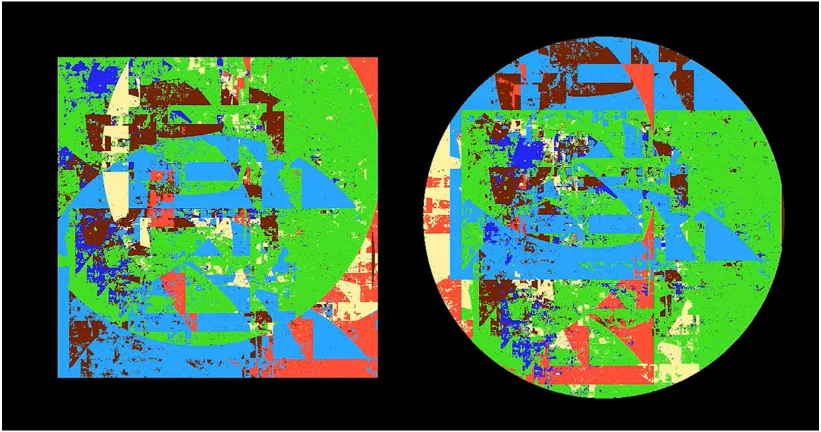
Example: Shift

Theorem (Kechris-Solecki-Todorčević)

Let G be a Borel subgraph of G_S. Then $\chi_B(G_S) \in \{1, 2, 3, \aleph_0\}$.

Why do we care: Paradoxical decompositions

equidecomposition $\iff$ matching



picture by András Máthé

Theorem (Grabowski-Máthé-Pikhurko, Marks-Unger)

Circle squaring with measurable and Borel pieces.

Why do we care: Complexity

Theorem (Todorčević-V)

Deciding Borel n-colorability is Σ^1_2-hard, for $n \geq 3$. In fact, deciding for a Borel subgraph G of $G_{\mathcal{S}}$ whether its Borel chromatic number ≤ 3 is Σ^1_2-hard.

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Theorem (Carroy-Miller-Schrittesser-V)

Borel 2-coloring is Π^1_1.

II. CSPs

Homomorphism problems

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4. n-SAT, HORN-SAT, etc.

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3Lin$_p$		GE

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$(*)$: "H has a nontrivial higher-dimensional symmetry/polymorphism"

III. Borel CSPs

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Theorem (Thornton)

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Theorem (Thornton)

If $(*)$ does not hold for H, then Borel H-HOM is Σ^1_2-hard.

Next step: Borel $3\mathrm{Lin}_p$.

Theorem (Grebík-V)

Borel $3LIN_p$ is Σ^1_2-hard.

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Conjecture

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The above is also equivalent to H having no non-trivial Borel polymorphisms.

Thank you for your attention!

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Theorem (Kátay-Tóth-V)

(ZF) Assume that H does not satisfy (*). Then $K_H \iff UL$.

Theorem (Kátay-Tóth-V)

(ZF) There exists a model of ZF, in which for every H the statement K_H holds iff H has (*).