Complexity Classification Transfer for CSPs via Algebraic Products

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Outline

1. Constraint satisfaction problems

2. Open problems from complexity of spatial reasoning
   - *n*-dimensional Cardinal Direction Calculus
   - *n*-dimensional Block Algebra

3. Classification of CSPs of first-order expansions of $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$

Constraint Satisfaction Problems

(relational) structure $\mathfrak{A} = (A; R^{\mathfrak{A}} : R \in \tau)$; finite signature τ

Definition (CSP)

$\mathfrak{B}$ – τ-structure

Constraint Satisfaction Problem for $\mathfrak{B}$ (CSP($\mathfrak{B}$)):

Input: finite τ-structure $\mathfrak{A}$

Question: Is there a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$?

Example: complete graph on 3 vertices

$$K_3 = (\{0, 1, 2\}; \neq)$$

CSP(K_3) = 3-colorability problem for graphs

more generally: CSP(K_n) = n-colorability problem

Complexity dichotomy

Theorem (Bulatov (2017), Zhuk (2017))

For every finite structure $\mathfrak{B}$ *with finite signature,* $\mathrm{CSP}(\mathfrak{B})$ *is in P or NP-complete.*

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τ-structure $\mathfrak{B}$ is:

- finitely bounded if there exists a universal τ-sentence ϕ such that a finite structure $\mathfrak{A}$ embeds into $\mathfrak{B}$ iff $\mathfrak{A} \models \phi$
- homogeneous if every isomorphism between finite substructures of $\mathfrak{B}$ can be extended to an automorphism of $\mathfrak{B}$

Conjecture (Bodirsky, Pinsker (2011))

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In the scope: fo-expansions of (algebraic powers of) $(\mathbb{Q}; <)$
$\rightarrow$ applications in temporal and spatial reasoning

Cardinal Direction Calculus

$\mathfrak{C} = (\mathbb{Q}^2; \mathrm{N}, \mathrm{E}, \mathrm{S}, \mathrm{W}, \mathrm{NE}, \mathrm{SE}, \mathrm{SW}, \mathrm{NW})$ (North, East, etc.)

N	E	S	W	NE	SE	SW	NW
$(=, >)$	$(>, =)$	$(=, <)$	$(<, =)$	$(>, >)$	$(>, <)$	$(<, <)$	$(<, >)$

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Ord-Horn formula: A conjunction of clauses of the form

$$x_1 \neq y_1 \vee \cdots \vee x_m \neq y_m \vee z_1 \circ z_0, \text{ where } \circ \in \{<, \leq, =\}.$$

Theorem (Ligozat (1998)): CSP of a reduct of CDC that contains the basic relations is in P if all relations can be defined by Ord-Horn formulas, and is NP-hard otherwise.

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natural generalization: CDC_n with the domain $\mathbb{Q}^n$
CDC conjecture (Balbiani, Condotta (2002)): The theorem also holds for the n-dimensional case.

Complexity of CDC

primitive positive formula: $\exists y_1, \ldots, y_l(\psi_1 \wedge \cdots \wedge \psi_m)$, ψ_i atomic formulas
example: $\phi(x, y) = \exists z\ R(x, y, z) \wedge R(x, x, z)$

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- denote $(<, \top)$ by $<_1$ and similarly for $=_1, <_2, =_2$
- $<_1, =_1, <_2, =_2$ are definable in $\mathfrak{C}$ by a pp-formula, e.g.

$$x <_1 y \Leftrightarrow \exists z \big(x(\mathrm{SW})z \wedge z(\mathrm{NW})y \big)$$

- $\mathrm{N}, \dots, \mathrm{NW}$ are definable in $(\mathbb{Q}^2; <_1, =_1, <_2, =_2)$ by a pp-formula

Complexity of CDC

Proposition (Jeavons (1998))

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures with the same domain. If every relation of $\mathfrak{A}$ has a pp-definition in $\mathfrak{B}$, then there is a poly-time reduction from $\mathrm{CSP}(\mathfrak{A})$ *to* $\mathrm{CSP}(\mathfrak{B})$.

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- fo-expansions of $\mathfrak{C}$ are primitively positively interdefinable with fo-expansions of $(\mathbb{Q}^2; <_1, =_1, <_2, =_2)$
  $\rightarrow$ their CSPs have the same complexity
- we prove the CDC conjecture by classifying fo-expansions of $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$

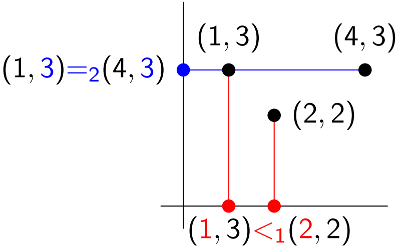
Algebraic products

Definition (algebraic product)

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be structures with signatures τ_1 and τ_2, respectively. The algebraic product $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ is the structure with the domain $A_1 \times A_2$ which has the following relations:

- for every $R \in \tau_1 \cup \{=\}$, the relation $R_1 = (R, \top)$,
- for every $R \in \tau_2 \cup \{=\}$, the relation $R_2 = (\top, R)$.

Example: $(\mathbb{Q}; <) \boxtimes (\mathbb{Q}; <) = (\mathbb{Q}^2; <_1, =_1, <_2, =_2)$



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$\longrightarrow$ natural generalization to n-fold algebraic products

Observation: Complexity classification of CSPs of fo-expansions of

$$\underbrace{(\mathbb{Q}; <) \boxtimes \cdots \boxtimes (\mathbb{Q}; <)}_{n} = (\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$$

leads to classification for CDC$_n$!

Primitive positive interpretations

generalizing pp-definitions → more applications

Definition (pp-interpretation)

Primitive positive interpretation of $\mathfrak{C}$ in $\mathfrak{B}$:
a *partial surjection* I from B^d to C (for some d) such that for every k-ary *relation* R defined by an *atomic formula* in $\mathfrak{C}$, $I^{-1}(R)$ as a dk-ary *relation* over B is definable in $\mathfrak{B}$ by a *pp-formula*.

Example: closed intervals $[a, b]$ over $\mathbb{Q}$ are elements of $\mathbb{Q}^2$ such that $a < b$

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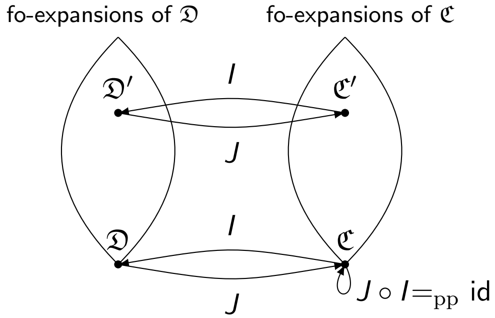
Proposition (folklore)

If $\mathfrak{C}$ has a pp-interpretation in $\mathfrak{B}$, then there is a poly-time reduction from CSP($\mathfrak{C}$) *to* CSP($\mathfrak{B}$).

Complexity classification transfer

- I – pp-interpretation of $\mathfrak{D}$ in $\mathfrak{C}$
- J – pp-interpretation of $\mathfrak{C}$ in $\mathfrak{D}$
- $J \circ I$ is pp-homotopic to the identity interpretation of $\mathfrak{C}$ (i.e., $\{(\bar{x}, \bar{y}) \mid J \circ I(\bar{x}) = \bar{y}\}$ is pp-definable in $\mathfrak{C}$)



$\Rightarrow$ for every fo-expansion $\mathfrak{C}'$ of $\mathfrak{C}$ there is an fo-expansion $\mathfrak{D}'$ of $\mathfrak{D}$ such that CSP($\mathfrak{C}'$) and CSP($\mathfrak{D}'$) are poly-time equivalent

Allen's Interval Algebra and Block Algebra

Allen's Interval Algebra:

- $\mathbb{I} = \{(a, b) \in \mathbb{Q}^2 \mid a < b\}$ – closed intervals
- 13 basic relations correspond to relative positions of intervals, e.g.:

s(X,Y):	`XXX`	f(X,Y):	`   XXX`	m(X,Y):	`XXXX`
starts	`YYYYYY`	*finishes*	`YYYYYY`	*meets*	`    YYYY`

- all relations: unions of basic relations

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Block Algebra (BA):

- domain: $\mathbb{I}^n$
- basic relations: n-tuples of Allen's basic relations
- all relations: unions of basic relations

Known results and open problems

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  $\rightarrow$ such a CSP is in P if all its relations are definable by Ord-Horn formulas and NP-hard otherwise
- Krokhin, Jeavons, Jonsson (2003): complexity classification for the CSPs for all subsets of Allen's relations
- **BA conjecture** (Balbiani, Condotta, del Cerro (2002)): The set of Ord-Horn relations is the unique maximal tractable subset of the block algebra that contains the basic relations.

Complexity classification transfer for Block Algebras

- Block Algebra with the basic relations is pp-interpretable in $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$ and vice versa
for $n = 2$:

$$I : (\mathbb{Q}^2)^2 \to \mathbb{I}^2,\ a <_1 b, a <_2 b$$
$$I((a_1, a_2), (b_1, b_2)) = ((a_1, b_1), (a_2, b_2))$$
$$J : \mathbb{I}^2 \to \mathbb{Q}^2$$
$$J((p_1, p_2), (q_1, q_2)) = (p_1, q_1)$$

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- all relations are fo-definable in basic relations
- the interpretations satisfy the assumptions for complexity classification transfer
- we prove the BA conjecture by transferring the classification of fo-expansions of $(\mathbb{Q}^n; <_1, =_1, \dots, <_n, =_n)$

Polymorphisms

Definition (polymorphism)

An operation $f : A^k \to A$ is a polymorphism of (or preserves) a structure $\mathfrak{A}$ if for every relation R of $\mathfrak{A}$ and for all tuples $\bar{r_1}, \ldots, \bar{r_k} \in R$ also $f(\bar{r_1}, \ldots, \bar{r_k}) \in R$ (computed row-wise).
Pol($\mathfrak{A}$) – the set of all polymorphisms of $\mathfrak{A}$

Example: $+$ is a polymorphism of $(\mathbb{Q}; <)$

$$\begin{pmatrix} 1 \\ \wedge \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ \wedge \\ 3 \end{pmatrix} \begin{matrix} \to \\ \\ \to \end{matrix} \begin{pmatrix} 3 \\ \wedge \\ 8 \end{pmatrix}$$

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Theorem (Bodirsky, Nešetřil (2006))

A relation $R \subseteq A^l$ is preserved by all polymorphisms of an ω-categorical structure $\mathfrak{A}$ iff R is pp-definable in $\mathfrak{A}$.

Properties of algebraic products

- $\mathfrak{A}_1$, $\mathfrak{A}_2$ homogeneous $\Rightarrow \mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ homogeneous
- $\mathfrak{A}_1$, $\mathfrak{A}_2$ ω-categorical $\Rightarrow \mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ ω-categorical

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- $\mathrm{Pol}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2) = \mathrm{Pol}(\mathfrak{A}_1) \times \mathrm{Pol}(\mathfrak{A}_2)$
- more generally: fo-expansions of $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$ contain the relations $=_i$
  $\Rightarrow$ all polymorphisms are of the form (f_1, f_2), $f_i \in \mathrm{Pol}(\mathfrak{A}_i)$

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Observation: $\mathrm{CSP}(\mathfrak{A}_1)$, $\mathrm{CSP}(\mathfrak{A}_2)$ in P $\Rightarrow \mathrm{CSP}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2)$ in P

Proof: Given input $\mathfrak{A}$ for $\mathrm{CSP}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2)$, run the algorithm for $\mathrm{CSP}(\mathfrak{A}_i)$ on the respective reducts of $\mathfrak{A}$.

Complexity of CSPs of (fo-expansions) of alg. products

$\mathfrak{A}_1, \mathfrak{A}_2$ – countable ω-categorical structures
$\theta_i : \mathrm{Pol}(\mathfrak{A}_1 \boxtimes \mathfrak{A}_2) \to \mathrm{Pol}(\mathfrak{A}_i)$ (projects on the i-th coordinate)

Follows from the results by Barto, Opršal, Pinsker (2018):

Proposition

Let $\mathfrak{D}$ be an fo-expansion of $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$. Let i be such that $\theta_i(\mathrm{Pol}(\mathfrak{D}))$ has a uniformly continuous minor-preserving (UCMP) map to $\mathrm{Pol}(K_3)$. Then $\mathrm{Pol}(\mathfrak{D})$ has a UCMP map to $\mathrm{Pol}(K_3)$ as well and $\mathrm{CSP}(\mathfrak{D})$ is NP-hard.

$\to$ $\mathrm{CSP}(\mathfrak{D})$ computationally hard in one coordinate implies $\mathrm{CSP}(\mathfrak{D})$ computationally hard!

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Question: If $\mathrm{CSP}(\mathfrak{D})$ is tractable in both coordinates, is then $\mathrm{CSP}(\mathfrak{D})$ tractable?

Tractable algebraic products

Finite-domain case:

- cyclic operation is an operation satisfying the identity

$$c(x_1, \ldots, x_k) = c(x_2, x_3, \ldots, x_k, x_1)$$

- $\theta_i(Pol(\mathfrak{D}))$ does not have an UCMP map to $\mathrm{Pol}(K_3)$
  $\Rightarrow \exists c_i \in \mathrm{Pol}(\mathfrak{D})$ such that $\theta_i(c_i)$ is cyclic (Barto, Kozik (2012))

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- then $\mathrm{Pol}(\mathfrak{D})$ contains the cyclic operation

$$c_1(c_2(x_1, \ldots, x_k), c_2(x_2, \ldots, x_k, x_1), \ldots, c_2(x_k, x_1, \ldots, x_{k-1}))$$

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Powers of $(\mathbb{Q}; <)$:

- a candidate polymorphism f – pseudo weak near unanimity (pwnu):

$$e_1(f(y, x, \ldots, x)) = e_2(f(x, y, \ldots, x)) = \cdots = e_k(f(x, \ldots, x, y)),$$

for some fixed $e_1, \ldots, e_k \in \mathrm{End}(\mathfrak{D})$

CSPs of fo-expansions of $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$

Theorem (Bodirsky, Kára (2009, 2010))

Let $\mathfrak{B}$ *be an fo-expansion of* $(\mathbb{Q}; <)$*. If* $\mathfrak{B}$ *contains a pwnu polymorphism, then* $\mathrm{CSP}(\mathfrak{B})$ *is in P. Otherwise,* $\mathrm{Pol}(\mathfrak{B})$ *has a uniformly continuous minor-preserving map to* $\mathrm{Pol}(K_3)$ *and* $\mathrm{CSP}(\mathfrak{B})$ *is NP-complete.*

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Theorem (Bodirsky, Jonsson, Martin, Mottet, S. (2022))

Let $\mathfrak{D}$ be an fo-expansion of $(\mathbb{Q}^n; <_1, =_1, \ldots, <_n, =_n)$. Exactly one of the following two cases applies:

- *$\theta_i(\mathrm{Pol}(\mathfrak{D}))$ contains a pwnu polymorphism for each i. In this case $\mathfrak{D}$ has a pwnu polymorphism and $\mathrm{CSP}(\mathfrak{D})$ is in P.*
- *There is i such that $\theta_i(\mathrm{Pol}(\mathfrak{D}))$ has a uniformly continuous minor-preserving map to $\mathrm{Pol}(K_3)$ and $\mathrm{CSP}(\mathfrak{D})$ is NP-complete.*

Corollaries of the classification

Using syntactic descriptions of the tractable cases in $(\mathbb{Q}; <)$ from (Bodirsky, Kára (2010)) and (Bodirsky, Chen, Wrona (2014)) we obtain:

Corollary

Suppose that $\mathfrak{D}$ has a binary signature. Exactly one of the following two cases applies:

- *Each relation in $\mathfrak{D}$ has an Ord-Horn definition (viewed as a relation of arity $2n$ over $\mathbb{Q}$) and $\mathrm{CSP}(\mathfrak{D})$ is in P.*
- $\mathrm{Pol}(\mathfrak{D})$ *has a UCMP map to* $\mathrm{Pol}(K_3)$ *and* $\mathrm{CSP}(\mathfrak{D})$ *is NP-complete.*

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Corollary

The CDC conjecture holds for every $n \geq 2$.

Corollary

The BA conjecture holds for every $n \geq 1$.

Proof idea for $n = 2$

NP-complete:

- follows directly from the previous proposition

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P:

- relations of $\mathfrak{D}$ are defined by fo-formulas in $<_i$ and $=_i$
- we may assume quantifier-free definitions in conjunctive normal form

Proof idea for $n = 2$

NP-complete:

- follows directly from the previous proposition

P:

- relations of $\mathfrak{D}$ are defined by fo-formulas in $<_i$ and $=_i$
- we may assume quantifier-free definitions in conjunctive normal form
- special clauses: *i*-determined (contain only relations with index i)
- under the assumptions we may restrict to conjunctions of weakly *i*-determined clauses, i.e.

$$\psi \vee \bigvee_{k \in \{1,\ldots,n\}} x_k \neq_j y_k,$$

where ψ is *i*-determined, $j \neq i$

Proof idea for $n = 2$

- if all clauses are i-determined, we can run the poly-time algorithms on 1-determined and 2-determined constraints separately
- such poly-time algorithms exist by the theorem for $(\mathbb{Q}; <)$

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Proposition

Let $\mathfrak{D}$ be an fo-expansion of $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$. TFAE:

1. *Every relation of $\mathfrak{D}$ has a definition by a conjunction of clauses each of which is either 1-determined or 2-determined.*
2. $\mathrm{Pol}(\mathfrak{D}) = \theta_1(\mathrm{Pol}(\mathfrak{D})) \times \theta_2(\mathrm{Pol}(\mathfrak{D}))$.
3. $\mathrm{Pol}(\mathfrak{D})$ *contains* (π_1^2, π_2^2).

$\rightarrow$ we might have also clauses that are not i-determined

Proof idea for $n = 2$

- sketch of the algorithm for weakly 1-determined clauses (oversimplified):
  1. compute pairs of variables (x, y) that satisfy $x =_2 y$ in all solutions to 2-determined constraints
  2. modify the weakly 1-determined clauses to obtain 1-determined constraints
  3. solve the 1-determined constraints by the poly-time algorithm from classification for $(\mathbb{Q}; <)$

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Question: Is there a polymorphism characterization of relations definable by weakly i-determined clauses?

What is next

Confirm the Bodirsky-Pinsker conjecture for:

- CSPs of fo-expansions of $\mathfrak{B} \boxtimes (\mathbb{Q}; <)$, where $\mathfrak{B}$ is a finite structure
- more generally: CSPs of structures fo-interpretable over $(\mathbb{Q}; <)$

Assuming the Bodirsky-Pinsker conjecture:

- classify complexity of CSPs of fo-expansions of $\mathfrak{A}_1 \boxtimes \mathfrak{A}_2$, where $\mathfrak{A}_i$ is finitely bounded homogeneous
  (remains the "tractable in both coordinates" case!)

Thank you for your attention