Conjugacy and Least Commutative Congruences in Semigroups

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Primary Conjugacy

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Let $\sim_p$ denote the transitive closure of the relation $\sim_p^1$. That is, $s \sim_p t$ if there exist $p_1, r_1, p_2, r_2, \ldots, p_n, r_n \in S^1$ such that

$$s = p_1 r_1, \; r_1 p_1 = p_2 r_2, \; r_2 p_2 = p_3 r_3, \ldots, \; r_{n-1} p_{n-1} = p_n r_n, \; r_n p_n = t.$$

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- In the C^*-algebra literature, $\sim_p^1$ is known as the *Murray-von Neumann equivalence*, and is used on projections in to construct the K_0-group.
- In symbolic dynamics, $\sim_p^1$ is called the *elementary shift equivalence*, and $\sim_p$ is called the *strong shift equivalence*.

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- Shoda–Albert–Muckenhoupt Theorem: Given a field F, a matrix in $\mathbb{M}_n(F)$ has trace 0 if and only if it is a commutator.
- It can be shown that for all $S, T \in \mathbb{M}_n(F)$, we have $\mathrm{trace}(S) = \mathrm{trace}(T)$ if and only if $S \sim_p T$.

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- If S is a group, and $s, t \in S$, then there exist $p, r \in S$ such that $s = pr$ and $rp = t$ if and only if there exist $p, r \in S$ such that $s = ptp^{-1}$ and $t = rsr^{-1}$.

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- Class Equation Theorem: Let G be a finite group, and let $g_1, \ldots, g_n \in G$ be representatives of the distinct conjugacy classes of G, not contained in the center $Z(G)$. Then

$$|G| = |Z(G)| + \sum_{i=1}^{n} [G : C(g_i)],$$

where $C(g) = \{h \in G \mid hg = gh\}$ for all $g \in G$.

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- For semigroups, $\sim_p$ is the most common generalization of conjugacy.

Other Notions of Conjugacy in Semigroups 1

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- If S has a zero element 0, then $s \cdot 0 = 0 \cdot t$ and $0 \cdot s = t \cdot 0$, for all $s, t \in S$, making the relation $\sim_o$ is universal.

Other Notions of Conjugacy in Semigroups 2

Definition

Let S be a semigroup and $s, t \in S$. Write $s \sim_c t$ if there exist $p \in \mathbb{P}(s)$ and $r \in \mathbb{P}(t)$ such that

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- This is a version of $\sim_o$ that can be useful even in semigroups with zero.

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Let S be a semigroup and $s, t \in S$. Write $s \sim_n t$ if there exist $p, r \in S^1$ such that

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- The relations $\sim_p$, $\sim_o$, and $\sim_c$ generally do not coincide with $\sim_i$ in an inverse semigroup.

Other Notions of Conjugacy in Semigroups 4

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Let S be a semigroup and $s, t \in S$. Write $s \sim_w t$ if there exist $p, r \in S^1$ and $m \in \mathbb{Z}^+$ such that

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- In that context, $\sim_p^1$ $(\exists p, r \in S^1\ (s = pr, rp = t))$ is called the *elementary shift equivalence*, and $\sim_p$ (the transitive closure of $\sim_p^1$) is called the *strong shift equivalence*.

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- Certain equivalence relations on directed graphs, studied in symbolic dynamics, when translated to rectangular matrices with nonnegative integer entries, correspond precisely to $\sim_p^1$ and $\sim_p$.
- More explicitly, two matrices S and T (possibly of different sizes), with entries from $\mathbb{N}$, are *elementary shift equivalent* if there exist rectangular matrices P and R (over $\mathbb{N}$) such that $S = PR$ and $RP = T$.

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- Each rectangular matrix in $\mathbb{N}$ can be viewed as an element of $\mathbb{M}_\infty(\mathbb{N})$, and then two such matrices S, T are elementary shift equivalent, respectively, strong shift equivalent, if and only if $S \sim_p^1 T$, respectively, $S \sim_p T$, as elements of $\mathbb{M}_\infty(\mathbb{N})$.

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- Proof: Suppose that $s, t \in S$ satisfy $s \sim_p^1 t$. Then $s^1 = pr$ and $rp = t^1$ for some $p, r \in S^1$. So $sp = prp = pt$ and $rs = rpr = tr$. Thus $s \sim_w t$. The inclusion $\sim_p \subseteq \sim_w$ then follows from the transitivity of $\sim_w$.

Kim/Roush Example

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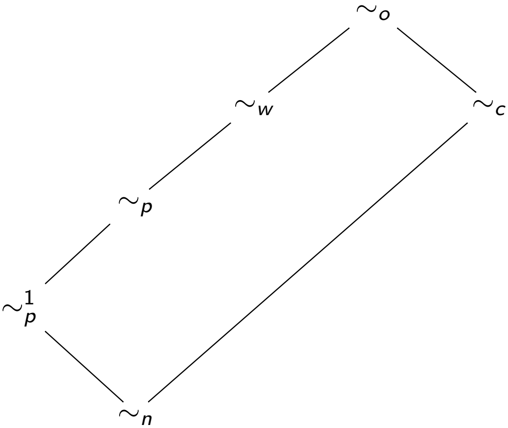
- In 1997, Kim and Roush showed that $\sim_p \neq \sim_w$ in $\mathbb{M}_\infty(\mathbb{N})$.
- Specifically, $S \sim_w T$, but $S \not\sim_p T$, for the matrices below:

$$S = \begin{pmatrix} 0 & 0 & 1 & 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 10 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 & 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 5 & 6 & 3 & 10 & 0 & 0 \\ 4 & 5 & 6 & 3 & 0 & 1 & 0 \end{pmatrix}.$$

Comparison of Semigroup Conjugacy Relations



$$s \sim_o t \iff \exists p, r \in S^1 \ (sp = pt, rs = tr)$$
$$s \sim_w t \iff \exists p, r \in S^1 \ \exists m \in \mathbb{Z}^+ \ (sp = pt, rs = tr, pr = s^m, rp = t^m)$$
$$s \sim_c t \iff \exists p \in \mathbb{P}(s) \ \exists r \in \mathbb{P}(t) \ (sp = pt, rs = tr)$$
$$s \sim_p^1 t \iff \exists p, r \in S^1 \ (s = pr, rp = t)$$
$$s \sim_n t \iff \exists p, r \in S^1 \ (sp = pt, rs = tr, rsp = t, ptr = s)$$

Permutation Conjugacy

Definition

Let S be a semigroup and $s, t \in S$. Write $s \sim_s^1 t$ if there exist $n \in \mathbb{Z}^+$, $p_1, \ldots, p_n \in S^1$, and $f \in \mathcal{S}(\{1, \ldots, n\})$, the symmetric group on $\{1, \ldots, n\}$, such that

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- Somewhat similar relations on semigroups have been studied before (e.g., Clifford/Cummings/Teymouri (2011), Piochi (1987)).

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- Clearly, $\sim_p^1 \subseteq \sim_s^1 \subseteq \sim_s$ and $\sim_p^1 \subseteq \sim_p \subseteq \sim_s$ in any semigroup.

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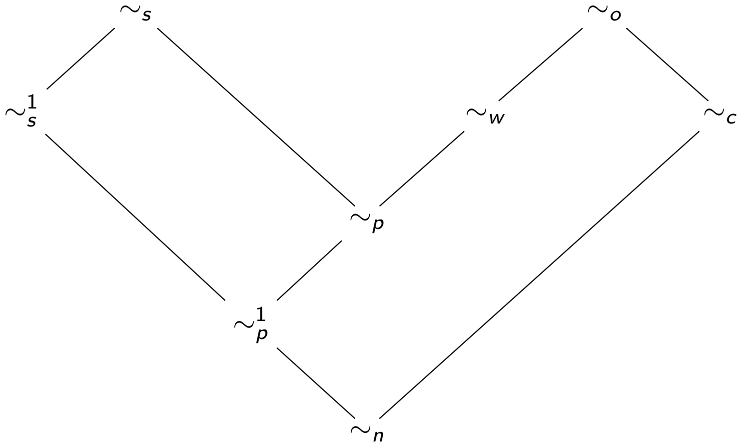

Permutation Conjugacy Properties

- Recall that given a semigroup S, an equivalence relation $\rho \subseteq S \times S$ is a *congruence* if $s \rho t$ implies that $(sr)\rho(tr)$ and $(rs)\rho(rt)$ for all $r, s, t \in S$.

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Let S be a semigroup, and let $\approx$ denote any of $\sim_p^1$, $\sim_p$, $\sim_s^1$. Then $\sim_s$ is the congruence generated by $\approx$, and it is the least congruence ρ on S such that S/ρ is commutative.

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Proposition

Let S be the free semigroup on a nonempty set Ω, and $\approx$ a reflexive symmetric relation on S such that $\approx \subseteq \sim_s$. Then the congruence generated by $\approx$ is $\sim_s$ if and only if $\alpha\beta \approx \beta\alpha$ for all $\alpha, \beta \in \Omega$.

Proof of Theorem

- Suppose that $s \sim_s^1 t$ for some $s, t \in S$, and let $r = p_{n+1} \in S$. Write $s = p_1 \cdots p_n$, $t = p_{f(1)} \cdots p_{f(n)}$ for some $p_1, \ldots, p_n \in S^1$ and $f \in \mathcal{S}(\{1, \ldots, n\})$. Then $sr = p_1 \cdots p_n p_{n+1}$ and $tr = p_{f(1)} \cdots p_{f(n)} p_{n+1}$, and so $sr \sim_s^1 tr$. Analogously, $rs \sim_s^1 rt$.

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- Now suppose that $s \sim_s t$ for some $s, t \in S$. Then there exist $q_1, \ldots, q_m \in S$ such that

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- Let ρ be a congruence on S such that $\sim_p^1 \subseteq \rho$, and let $[s]_\rho$ denote the ρ-congruence class of $s \in S$. Then for all $s, t \in S$ we have $st \sim_p^1 ts$, and hence

$$[s]_\rho [t]_\rho = [st]_\rho = [ts]_\rho = [t]_\rho [s]_\rho.$$

  Therefore S/ρ is commutative. In particular, $S/\sim_s$ is commutative.

Proof of Theorem (Continued)

- Suppose that ρ is a congruence on S such that S/ρ is commutative. Then for all $p_1, \ldots, p_n \in S^1$ and $f \in \mathcal{S}(\{1, \ldots, n\})$,

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- Finally, let ρ denote the congruence on S generated by $\sim_p^1$, $\sim_p$, or $\sim_s^1$. Since $\sim_p^1 \subseteq \rho$, the quotient S/ρ is commutative, and so $\sim_s \subseteq \rho$. But since $\sim_p^1, \sim_p, \sim_s^1 \subseteq \sim_s$, we have $\sim_s = \rho$.

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Proposition (Generalizing Leroy/Nasernejad)

Let S be a semigroup and $s, t \in S$. Write $s \sim_*^1 t$ if there exist $p_1, p_2, p_3 \in S^1$ such that

$$s = p_1 p_2 p_3, \;\; p_1 p_3 p_2 = t,$$

and denote by $\sim_*$ the transitive closure of the relation $\sim_*^1$. Then $\sim_* = \sim_s$.

Permutation Conjugacy in Groups

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Let G be a group and $s, t \in G$. Then $st^{-1} \in [G, G]$ (the multiplicative commutator subgroup) if and only if $s \sim_s^1 t$ if and only if $s \sim_s t$.

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for some $p_i, r_i \in G$. Then

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showing that $s \sim_s^1 t$. Clearly, $s \sim_s^1 t$ implies that $s \sim_s t$.

Now, let $\phi : G \to G/[G, G]$ be the natural projection. Since $G/[G, G]$ is commutative, $\sim_s \subseteq \ker(\phi) = [G, G]$, by the theorem. That is, if $s \sim_s t$, then $st^{-1} \in [G, G]$. $\square$

Permutation Conjugacy in Rings

Corollary

Let R be a ring, let I_1 be the additive subgroup of R generated by $\{s - t \mid s, t \in R, s \sim_s^1 t\}$, and let I_2 be the additive subgroup of R generated by $\{s - t \mid s, t \in R, s \sim_s t\}$. Then $I_1 = I_2 = [R, R]$, where $[R, R]$ is the ideal of R generated by the additive commutators ($[p, r] = pr - rp$).

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$R/[R, R]$ is a commutative ring, and hence a commutative semigroup. Thus, by the theorem, if $s \sim_s t$, for some $s, t \in R$, then $s - t \in [R, R]$. In particular, $I_1 \subseteq I_2 \subseteq [R, R]$.

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As an additive group, $[R, R]$ is generated by elements of the form

$$q(rs - sr)t = qrst - qsrt.$$

Since $qrst \sim_s^1 qsrt$, we have $[R, R] \subseteq I_1$, and so $[R, R] = I_1 = I_2$. $\square$

Closure Under the Permutation Conjugacy

Proposition

Let S be a semigroup, $T \subseteq S$, and $\overline{T} \subseteq S$ the closure of T under $\sim_s$.

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4. If S is an inverse semigroup, and T is an inverse subsemigroup of S, then so is $\overline{T}$.
5. If S is a group, and T is a subgroup of S, then $\overline{T}$ is the (normal) subgroup of S generated by T and $[S, S]$.

Classical Transformation Semigroups

For a set Ω, denote by $\mathcal{T}(\Omega)$ the monoid of all functions $\Omega \to \Omega$, by $\mathcal{PT}(\Omega)$ the monoid of all partial functions $\Omega \to \Omega$, and by $\mathcal{I}(\Omega)$ the symmetric inverse monoid on Ω.

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Definition

Let Ω be a set, $\Sigma \subseteq \Omega$ nonempty, and $s \in \mathcal{T}(\Omega)$. We say that Σ is a *connected component* or *cycle* of s if the following conditions are satisfied:

(i) $s(\alpha) \in \Sigma$ if and only if $\alpha \in \Sigma$, for all $\alpha \in \Omega$;

(ii) Σ has no proper nonempty subset satisfying (i).

Theorem (Kudryavtseva/Mazorchuk (2007))

Let Ω be a finite set, $S \in \{\mathcal{T}(\Omega), \mathcal{PT}(\Omega), \mathcal{I}(\Omega)\}$, and $s, t \in S$. For each $n \leq |\Omega|$, let l_n^s denote the number of cycles of s of size n, and let $\mathrm{ct}(s) = (l_1^s, \ldots, l_{|\Omega|}^s)$. Then $s \sim_p t$ if and only if $\mathrm{ct}(s) = \mathrm{ct}(t)$.

Kudryavtseva and Mazorchuk also gave a (much more complicated) description of $\sim_p$ in $\mathcal{I}(\Omega)$ for countable Ω.

Classical Transformation Semigroups

Proposition

Let Ω be a set. If Ω is infinite, then $\sim_s$ is the universal relation on $\mathcal{T}(\Omega)$, $\mathcal{PT}(\Omega)$, and $\mathcal{I}(\Omega)$. If Ω is finite, then in each of these semigroups there are three $\sim_s$-congruence classes–consisting of even permutations of Ω, odd permutations of Ω, and (partial) transformations with image of size $< |\Omega|$.

Classical Transformation Semigroups

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Proof Summary for $\mathcal{T}(\Omega)$: Finite Case.

- Suppose that Ω is finite, and consider $s, t \in \mathcal{S}(\Omega)$ (the permutation group of Ω) such that $st^{-1} \in [\mathcal{S}(\Omega), \mathcal{S}(\Omega)]$ (which is the alternating subgroup, by Ore's theorem (1951)). Then $s \sim_s t$.

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- According to Mal'cev's theorem (1979), there is only one non-universal congruence $\approx$ on $\mathcal{T}(\Omega)$ that relates all the odd permutations and all the even permutations (which also relates all the transformations in $\mathcal{T}(\Omega) \setminus \mathcal{S}(\Omega)$). So $\approx \subseteq \sim_s$.

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- But $\mathcal{T}(\Omega)/\approx$ is commutative, and so $\sim_s \subseteq \approx$. Hence $\sim_s = \approx$.

$\square$

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- By Ore's theorem (1951), there exist $r_1, r_2 \in \mathcal{S}(\Omega)$ such that $q = r_1 r_2 r_1^{-1} r_2^{-1}$, and so

$$p = sqt = s(r_1 r_2 r_1^{-1} r_2^{-1})t \sim_s (st)(r_1 r_1^{-1})(r_2 r_2^{-1}) = 1.$$

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- Thus the $\sim_s$-equivalence class of 1 is all of $\mathcal{T}(\Omega)$.

□

Injective Transformation Semigroup

For a set Ω, denote by $\mathcal{J}(\Omega)$ the monoid of all injective functions $\Omega \to \Omega$.

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Theorem (Generalizing Araújo/Kinyon/Konieczny/Malheiro)

Let Ω be a set and $s, t \in \mathcal{J}(\Omega)$. Then the following are equivalent.

1. $s \sim_p t$.
2. $s \sim_p^1 t$.
3. $s \sim_n t$.
4. $s = ptp^{-1}$ for some $p \in \mathcal{S}(\Omega)$.
5. There is a bijection between the set of cycles of s and the set of cycles of t, that sends each cycle to one of the same type.

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Theorem

Let Ω be a set and $s, t \in \mathcal{J}(\Omega)$. If Ω is finite, and hence $\mathcal{J}(\Omega) = \mathcal{S}(\Omega)$, then $s \sim_s t$ if and only if st^{-1} is an even permutation. If Ω is infinite, then $s \sim_s t$ if and only if $|\Omega \setminus s(\Omega)| = |\Omega \setminus t(\Omega)|$.

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For a set Ω, denote by $\mathcal{O}(\Omega)$ the monoid of all surjective functions $\Omega \to \Omega$.

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For any $s \in \mathcal{T}(\Omega)$, define $N(s) = \{\alpha \in \Omega \mid \exists \beta \in \Omega \setminus \{\alpha\} \ (s(\alpha) = s(\beta))\}$,

$$C(s) = \{\alpha \in \Omega \mid |s^{-1}(\alpha)| > 1\}, \text{ and } m(s) = \sup\{|s^{-1}(\alpha)| \mid \alpha \in \Omega\}.$$

We say that s *achieves* $m(s)$ if $m(s) = |s^{-1}(\alpha)|$ for some $\alpha \in \Omega$.

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Theorem

Let Ω be a countably infinite set, and $s, t \in \mathcal{O}(\Omega)$. Write $s \approx t$ if any of the following holds.

1. $|N(s)|, |N(t)| < \aleph_0$, and $|N(s)| - |C(s)| = |N(t)| - |C(t)|$.
2. $|N(s)| = |N(t)| = \aleph_0$, and $m(s), m(t) < \aleph_0$.
3. $m(s) = m(t) = \aleph_0$, but s and t do not achieve $m(s) = m(t)$.
4. $m(s) = m(t) = \aleph_0$, and s and t achieve $m(s) = m(t)$.

Then $\approx$ is a congruence, and $\sim_s \subseteq \approx$.

Rees Matrix Semigroups

Let G be a group, I and Λ nonempty sets, and $P = (p_{\lambda i})$ a $\Lambda \times I$ "sandwich" matrix with entries in $G \cup \{0\}$, such that no row or column consists entirely of zeros. Then $\mathcal{M}^0(G; I, \Lambda; P) = (I \times G \times \Lambda) \cup \{0\}$, with multiplication given by

$$(i, s, \lambda)(j, t, \mu) = \begin{cases} (i, s p_{\lambda j} t, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

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According to Rees's theorem, $\mathcal{M}^0(G; I, \Lambda; P)$ is *completely* 0-*simple* (i.e., it is a semigroup S such that $S^2 \neq \{0\}$, S and $\{0\}$ are the only ideals, and the inverse semigroup $E(S)$ of idempotents of S has an element minimal in the natural partial order $\leq$), and every completely 0-simple semigroup is of this form.

Rees Matrix Semigroups

Theorem

The following hold for all $(i, s, \lambda), (j, t, \mu) \in \mathcal{M}^0(G; I, \Lambda; P) \setminus \{0\}$.

1. $(i, s, \lambda) \sim_p 0 \iff p_{\lambda i} = 0$.
2. $(i, s, \lambda) \sim_p (j, t, \mu) \iff$ either $p_{\lambda i} = 0 = p_{\mu j}$, or $p_{\lambda i} \neq 0 \neq p_{\mu j}$ and $r p_{\lambda i} s = t p_{\mu j} r$ for some $r \in G$.

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3. If P has any 0 entries, then $(i, s, \lambda) \sim_s 0$.
4. If P has only nonzero entries, then $(i, s, \lambda) \not\sim_s 0$, and $(i, s, \lambda) \sim_s (j, t, \mu)$ $\iff st^{-1} \in H$, where H is the subgroup of G generated by $[G, G]$ and the entries of P.

Graph Inverse Semigroups

Let $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$ be a directed graph, with path set $\mathrm{Path}(E)$, and closed path set $\mathrm{ClPath}(E)$ (consisting of $p \in \mathrm{Path}(E)$ such that $\mathbf{s}(p) = \mathbf{r}(p)$).

The *graph inverse semigroup $G(E)$ of E* is the semigroup (with zero) generated by the vertex set E^0 and the edge set E^1, together with $\{e^{-1} \mid e \in E^1\}$, satisfying the relations:
(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$,
(E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$ for all $e \in E^1$,
(E2) $\mathbf{r}(e)e^{-1} = e^{-1}\mathbf{s}(e) = e^{-1}$ for all $e \in E^1$,
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Each nonzero element of $G(E)$ is of the form pq^{-1}, for some $p, q \in \mathrm{Path}(E)$, where $(e_1 \cdots e_n)^{-1} = e_n^{-1} \cdots e_1^{-1}$ for $e_1, \dots, e_n \in E^1$ and $v^{-1} = v$ for $v \in E^0$.

$G(E)$ is an inverse semigroup, with $(pq^{-1})^{-1} = qp^{-1}$ for all paths p, q. (I.e., for each $s \in G(E)$ there is a unique $t \in G(E)$ satisfying $sts = s$ and $tst = t$.)

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These semigroups were introduced by Ash/Hall (1975), in order to show that every partial order can be realized as that of the nonzero $\mathscr{J}$-classes of an inverse semigroup.

Graph Inverse Semigroups

Lemma

Let E be a graph and $p, q \in \mathrm{ClPath}(E)$. Write $p \approx q$ if there exist $r_1, r_2 \in \mathrm{Path}(E)$ such that $p = r_1 r_2$ and $r_2 r_1 = q$. Then $\approx$ is an equivalence relation on $\mathrm{ClPath}(E)$.

Theorem

Let E be a graph and $s, t \in G(E)$. Then $s \sim_p t$ if and only if exactly one of the following holds.

1. There exist $p_1, p_2 \in \mathrm{ClPath}(E)$ and $q, r \in \mathrm{Path}(E)$ such that $p_1 \approx p_2$, $\mathbf{r}(q) = \mathbf{s}(p_1)$, $\mathbf{r}(r) = \mathbf{s}(p_2)$, $s = qp_1q^{-1}$, and $t = rp_2r^{-1}$.
2. There exist $p_1, p_2 \in \mathrm{ClPath}(E) \setminus E^0$ and $q, r \in \mathrm{Path}(E)$ such that $p_1 \approx p_2$, $\mathbf{r}(q) = \mathbf{r}(p_1)$, $\mathbf{r}(r) = \mathbf{r}(p_2)$, $s = qp_1^{-1}q^{-1}$, and $t = rp_2^{-1}r^{-1}$.
3. Neither s nor t is of the form qpq^{-1} or $qp^{-1}q^{-1}$, for any $p \in \mathrm{ClPath}(E)$ and $q \in \mathrm{Path}(E)$. (This case occurs if and only if $s \sim_p 0 \sim_p t$.)

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Theorem

Let E be a graph and $s, t \in G(E)$. Then $s \sim_s t$ if and only if exactly one of the following holds.

1. There exists a vertex $v \in E^0$ such that $\mathbf{r}^{-1}(v) = \{v\}$ and $s = v = t$.
2. There exist a loop $e \in E^1$ (i.e., a closed path with only one edge) and $n_1, m_1, n_2, m_2 \in \mathbb{N}$ such that $\mathbf{r}^{-1}(\mathbf{s}(e)) = \{\mathbf{s}(e), e\}$, $s = e^{n_1}e^{-m_1}$, $t = e^{n_2}e^{-m_2}$, and $n_1 - m_1 = n_2 - m_2$.
3. Neither s nor t is of the previous two forms. (This case occurs if and only if $s \sim_s 0 \sim_s t$.)

Thank you!

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