Axiomatizing small varieties of periodic ℓ-pregroups

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Lattice-ordered groups

A **lattice-ordered group** (**ℓ-group**) is an algebra $\langle L, \wedge, \vee, \cdot, ^{-1}, 1\rangle$ s.t.

- $\langle L, \cdot, ^{-1}, 1\rangle$ is a group,
- $\langle L, \wedge, \vee\rangle$ is a lattice,
- $a(b \vee c)d = abd \vee acd$ and $a(b \wedge c)d = abd \wedge acd$ for all $a, b, c, d \in S$.

We denote by LG the variety of ℓ-groups.

▷ ℓ-groups have been extensively studied in the last century.

▶ Every ℓ-group has a distributive lattice reduct.

▶ (Holland 1963) Every ℓ-group embeds into the automorphism ℓ-group

$$\mathbf{Aut}(\mathbf{C}) = \langle \mathrm{Aut}(\mathbf{C}), \wedge, \vee, \circ, ^{-1}, id_C\rangle,$$

for some chain $\mathbf{C} = \langle C, \leq\rangle$, where $\wedge$ and $\vee$ are defined point-wise.

Pregroups

A **pregroup** is a structure $\mathbf{L} = \langle L, \cdot, 1, ^\ell, ^r, \leq \rangle$ such that $\langle L, \cdot, 1 \rangle$ is a monoid, $\langle L, \leq \rangle$ is a poset and for each $a \in L$,

$$a^\ell a \leq 1 \leq aa^\ell \text{ and } aa^r \leq 1 \leq a^r a.$$

- Pregroups where introduced in mathematical linguistics (Lambek, Buzskowski).

Lattice-ordered pregroups

A **lattice-ordered pregroup** (**ℓ-pregroup**) is an algebra $(L, \wedge, \vee, \cdot, ^{\ell}, ^{r}, 1)$ such that $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, 1)$ is a monoid, multiplication preserves the lattice order $\leq$, and for every $a \in L$,

$$a^{\ell}a \leq 1 \leq aa^{\ell} \text{ and } aa^{r} \leq 1 \leq a^{r}a.$$

- ▷ ℓ-pregroups are precisely the involutive residuated lattices, where multiplication coincides with its De Morgan dual.
  - ↳ In ℓ-pregroups: $(a \cdot b)^{\ell} = b^{\ell} \cdot a^{\ell}$, $(a^{\ell})^{r} = (a^{r})^{\ell} = a$, $a(b \bullet c)d = abd \bullet acd$ ($\bullet \in \{\wedge, \vee\}$), and De Morgan laws hold.

An ℓ-pregroup $\mathbf{L}$ is called **distributive** if its lattice reduct is distributive.

- ▷ Every ℓ-group 'is' a distributive ℓ-pregroup with $a^{\ell} = a^{r} = a^{-1}$.
- ▷ **Longstanding problem:** is every ℓ-pregroup distributive?
- ▶ ℓ-pregroups are semi-distributive (Galatos-Jipsen-Kinyon-Přenosil 2021).

An ℓ-pregroup is called ***n*-periodic** (for $n \geq 1$) if it satisfies the equation

$$x^{\ell^n} \approx x^{r^n} \text{(or equivalently } x^{\ell^{2n}} \approx x\text{)},$$

where $x^{\ell^1} = x^\ell$, $x^{\ell^{n+1}} = (x^{\ell^n})^\ell$ and similarly for x^{r^n}

▷ For $n = 1$, we get ℓ-groups.

We denote the variety of distributive ℓ-pregroups by DLP and the variety of n-periodic ℓ-pregroups by LP_n.

▶ Every periodic ℓ-pregroup is distributive, i.e., $\mathsf{LP}_n \subseteq \mathsf{DLP}$ for any $n \geq 1$ (Galatos-Jipsen 2012).

Symmetric ℓ-pregroup of a chain

Let $\mathbf{C}$ be a chain. Maps $f\colon C \to C$ and $g\colon C \to C$, are said to form a **residuated pair** if for all $a, b \in C$:

$$f(a) \leq b \iff a \leq g(b).$$

We say that g is the **residual** of f and f is the **dual residual** of g.

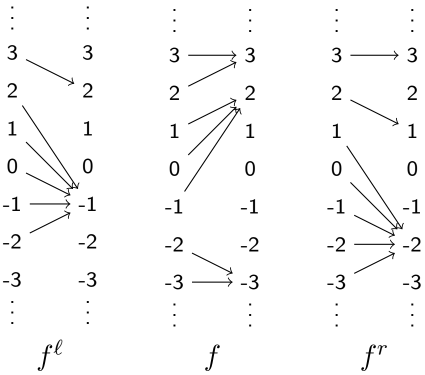
Fact. For any $f\colon C \to C$, the dual residual f^ℓ and residual f^r are unique if they exist: $f^\ell(y) = \bigwedge\{x : y \leq f(x)\}$ and $f^r(y) = \bigvee\{x : f(x) \leq y\}$.

The set $\mathrm{F}(\mathbf{C})$ of maps on C with residuals and dual residuals of every order forms a distributive ℓ-pregroup $\mathbf{F}(\mathbf{C}) = \langle \mathrm{F}(\mathbf{C}), \wedge, \vee, \circ, {}^\ell, {}^r, id_C \rangle$, where $\circ$ is composition, and $\wedge$ and $\vee$ are defined point-wise.

- Every distributive ℓ-pregroup embeds into $\mathbf{F}(\mathbf{C})$ for some chain $\mathbf{C}$ (Galatos-Jipsen 2012).

Example $\mathbf{F}(\mathbb{Z})$

The set $\mathrm{F}(\mathbb{Z})$ consists of all order-preserving functions from $\mathbb{Z}$ to $\mathbb{Z}$ that are **finite-to-one** (the preimage of an element is a finite set).



$$f^\ell(y) = \bigwedge\{x : y \le f(x)\}$$

$$f^r(y) = \bigvee\{x : f(x) \le y\}$$

$$f^\ell(3) = 2 \text{ and } f^r(3) = 3$$

$$f^\ell(2) = -1 \text{ and } f^r(2) = 1$$

$$f^\ell(1) = -1 = f^\ell(-2)$$

$$f^r(1) = -2 = f^r(-3)$$

- DLP $= \mathbb{V}(\mathbf{F}(\mathbb{Z}))$ (Galatos-Gallardo 2024).

Representation theorem in the periodic case

Fact. For an ℓ-pregroup $\mathbf{L}$ and $n \in \mathbb{Z}^+$ the set $\{a \in L : a^{\ell^n} = a^{r^n}\}$ of n-periodic elements of $\mathbf{L}$ forms an n-periodic subalgebra.

For a chain $\mathbf{C}$ and $n \in \mathbb{Z}^+$ we denote by $\mathbf{F}_n(\mathbf{C})$ the subalgebra of $\mathbf{F}(\mathbf{C})$ consisting of its n-periodic members. $\rightsquigarrow$ If $k \mid n$, then $\mathbf{F}_k(\mathbf{C}) \leq \mathbf{F}_n(\mathbf{C})$.

- Every n-periodic ℓ-pregroup embeds into $\mathbf{F}_n(\mathbf{C} \overrightarrow{\times} \mathbb{Z}) \cong \mathbf{Aut}(\mathbf{C}) \wr \mathbf{F}_n(\mathbb{Z})$ for some chain $\mathbf{C}$ (Galatos-Gallardo 2025).
  $\rightsquigarrow$ n-periodic maps are exactly the maps s.t. $f(c, x+n) = f(c,x) + n$.

- $\mathsf{DLP} = \bigvee_n \mathsf{LP}_n = \bigvee_n \mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ (Galatos-Gallardo 2025).

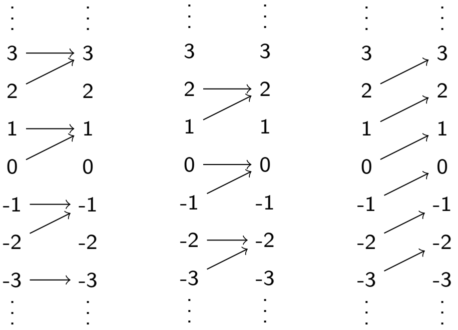
- $\mathbb{V}(\mathbf{F}_n(\mathbb{Z})) \subsetneq \mathsf{LP}_n = \mathbb{V}(\mathbf{F}_n(\mathbb{Q} \overrightarrow{\times} \mathbb{Z}))$ (Galatos-Gallardo 2025).

Quesition: How is $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ axiomatized? Is it finitely axiomatized?

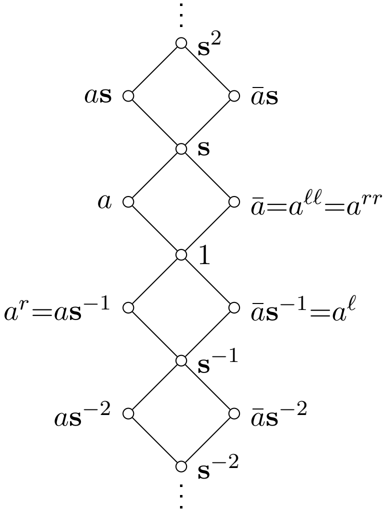
We will answer the second question positively.

Example $\mathbf{F}_2(\mathbb{Z})$

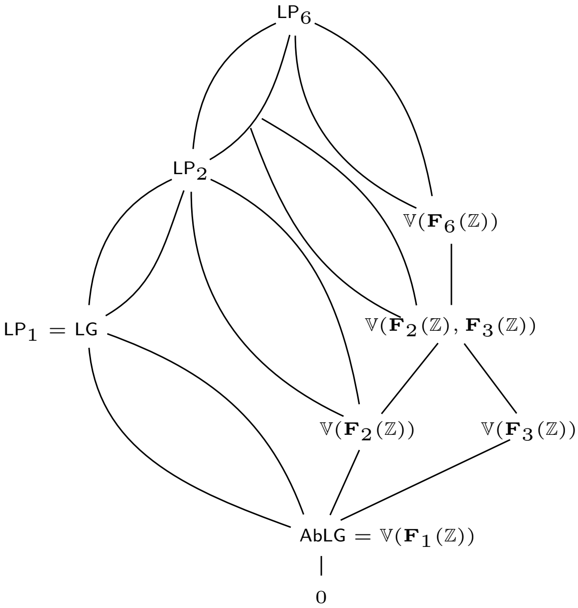
$\mathrm{F}_2(\mathbb{Z}) = \{a\mathbf{s}^n, \mathbf{s}^n, \bar{a}\mathbf{s}^n : n \in \mathbb{Z}\}$, where $\bar{a} = a^{\ell\ell}$.



a $\bar{a}$ $\mathbf{s}$



Part of the subvariety lattice of LP_6



Group skeleton

For any ℓ-pregroup $\mathbf{L}$, **the group skeleton $\mathbf{L}_g$** of $\mathbf{L}$ is the subalgebra of $\mathbf{L}$ with universe $L_g = \{a \in L : a^{\ell\ell} = a\}$.

- $\mathbf{L}_g$ is an ℓ-group and consists exactly of the invertible elements in $\mathbf{L}$.

Rmk. For an ℓ-pregroup $\mathbf{L}$ the map $a \mapsto a^{\ell\ell}$ is an automorphism:

- L_g is its set of fixed points.
- n-periodicity means that this automorphism has order n.

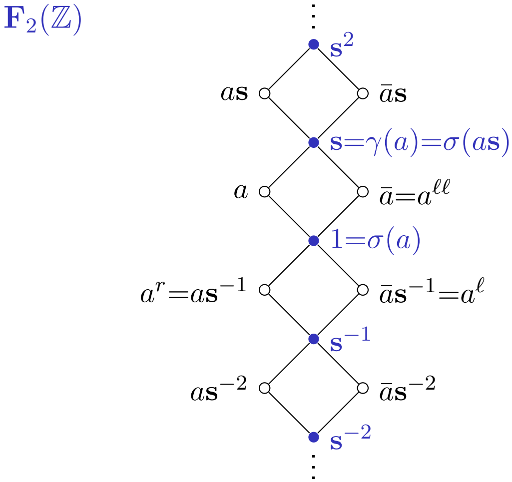
Let $\mathbf{L}$ be an n-periodic ℓ-pregroup. We define the terms

$$\sigma_n(x) := x \wedge x^{\ell\ell} \wedge \ldots \wedge x^{\ell^{2n-2}},$$
$$\gamma_n(x) := x \vee x^{\ell\ell} \vee \ldots \vee x^{\ell^{2n-2}}.$$

Lem. For any n-periodic ℓ-pregroup, $L_g = \sigma[L] = \gamma[L]$. The map $a \mapsto \sigma(a)$ is a conucleus on $\mathbf{L}$ and dually for the map $a \mapsto \gamma(a)$.

Group skeleton of $\mathbf{F}_n(\mathbb{Z})$

- The group skeleton of $\mathbf{F}_n(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and is generated by the element s with $\mathrm{s}(x) = x + 1$.



Groups skeleton and convex subalgebras

For any n-periodic ℓ-pregroup $\mathbf{L}$ and $a \in L$, we set $\|a\| := \sigma(a)^{-1} \vee \gamma(a)$.

For $S \subseteq L$ we denote by $\mathrm{C}_L(S)$ the convex subuniverse of $\mathbf{L}$ generated by S and we denote the lattice of convex subuniverses of $\mathbf{L}$ by $\mathcal{C}(\mathbf{L})$.

Lem. Let $\mathbf{L}$ be an n-periodic ℓ-pregroup and $S \subseteq L$. Then

$$\mathrm{C}_L(S) = \{a \in L : \|a\| \leq \|s_1\| \cdots \|s_k\| \text{ for some } s_1, \ldots, s_k \in S\}.$$

Thm. For any n-periodic ℓ-pregroups the lattices $\mathcal{C}(\mathbf{L})$ and $\mathcal{C}(\mathbf{L}_g)$ are isomorphic via the inverse maps $H \mapsto H \cap L_g$ and $K \mapsto \mathrm{C}_L(K)$.

Cor. For any n-periodic ℓ-pregroups, $\mathcal{C}(\mathbf{L})$ is a frame. In particular, it is a distributive lattice.

Convex normal subalgebras

A subset M of an ℓ-pregroup $\mathbf{L}$ is called **normal** if it is closed under conjugations, i.e., for $a \in M$ and $b \in L$, $bab^{\ell} \wedge 1 \in M$.

- This is equivalent to the usual definition for residuated lattices.

For an ℓ-pregroup $\mathbf{L}$, $\mathcal{N}(\mathbf{L})$ is the lattice of its normal convex subuniverses.

- $\mathcal{N}(\mathbf{L})$ is isomorphic to $\mathbf{Con}(\mathbf{L})$ (residuated lattice theory).

Lem. For any n-periodic ℓ-pregroup $\mathbf{L}$, $\mathcal{N}(\mathbf{L})$ embeds into $\mathcal{N}(\mathbf{L}_g)$ as a bounded infinitary meet-semilattice via the map $H \mapsto H \cap L_g$.

$\rightsquigarrow$ If $\mathbf{L}_g$ is (finitely) subdirectly irreducible or simple, then $\mathbf{L}$ is (finitely) subdirectly irreducible or simple.

Convex normal subalgebras continued

We can do better for algebras $\mathbf{L} \in \mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ because $\mathbf{F}_n(\mathbb{Z})$ satisfies $\sigma_n(x)^n y \approx y\sigma_n(x)^n$:

- For each $g \in L_g$ with $g \leq 1$ and $a \in L$, we have $g^n \leq aga^\ell \wedge 1$.

Lem. If $\mathbf{L}$ is an n-periodic ℓ-pregroup with $\mathbf{L} \models \sigma_n(x)^n y \approx y\sigma_n(x)^n$, then the lattices $\mathcal{N}(\mathbf{L})$ and $\mathcal{N}(\mathbf{L}_g)$ are isomorphic via the maps $H \mapsto H \cap L_g$ and $K \mapsto \mathrm{C}_L(K)$.

Fact (Darnel et al. 2016). An ℓ-group that satisfies $x^n y \approx yx^n$ is abelian.

Prop. Any n-periodic ℓ-pregroup $\mathbf{L}$ with $\mathbf{L} \models \sigma_n(x)^n y \approx y\sigma_n(x)^n$ is finitely subdirectly irreducible iff $\mathbf{L}_g$ is a totally ordered abelian ℓ-group.

Main result and strategy

Main Thm. The variety $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is axiomatized by $\sigma_n(x)^n y \approx y\sigma_n(x)^n$ and its FSIs are exactly the n-periodic ℓ-pregroups $\mathbf{L}$ with $\mathbf{L}_g$ a totally ordered abelian ℓ-group.

The proof of the main result can be split into three steps:

- Show that $\mathbf{F}_n(\mathbb{Z}) \models \sigma_n(x)^n y \approx y\sigma_n(x)^n$ and the FSIs of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ have a totally ordered abelian group skeleton (already discussed).
- Show that $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is closed under lexicographic products with totally ordered abelian ℓ-groups.
- Show that every finitely generated **proper** n-periodic ℓ-pregroup with a totally ordered abelian group skeleton is of the form $\mathbf{H}\overrightarrow{\times}\mathbf{F}_k(\mathbb{Z})$ with $k \mid n$ and $\mathbf{H}$ a finitely generated totally ordered abelian ℓ-group.

Lexicographic products

Fact. For a totally ordered group $\mathbf{G}$ and an (n-periodic) ℓ-pregroup $\mathbf{L}$ their lexicographic product $\mathbf{G} \overrightarrow{\times} \mathbf{L}$ is again an (n-periodic) ℓ-pregroup.

An element g of an ℓ-pregroup $\mathbf{L}$ is called a **strong order unit** if for each $a \in L$, there exists $n \in \mathbb{Z}^+$ such that $a \leq g^n$.

Lem. If an ℓ-pregroup $\mathbf{L}$ contains a central invertible strong order unit g, then $\mathbb{V}(\mathbf{L})$ is closed under lexicographic products with totally ordered abelian groups.

Sketch. Embed $\mathbb{Z} \overrightarrow{\times} \mathbf{L}$ into an ultrapower $\mathbf{L}^{\mathbb{N}}/U$ via $(1,0) \mapsto [(g, g^2, \dots)]$ and $(0,a) \mapsto [(a, a, \dots)]$ and check that this implies the closure under lex-products.

Cor. $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is closed under lexicographic products with totally ordered abelian group.

Proof. Consider $\mathbf{s}^n \in \mathrm{F}_n(\mathbb{Z})$, where $\mathbf{s}(x) = x + 1$.

Main result and strategy

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Wreath products

Let $\mathbf{J}$ be a chain and $\mathbf{L}$ be an ℓ-pregroup. The **wreath product** is defined as

$$\mathbf{Aut}(\mathbf{J}) \wr \mathbf{L} = (\mathrm{Aut}(\mathbf{J}) \times L^J, \wedge, \vee, \cdot, (id, \overline{1}), {}^{\ell}, {}^{r}),$$

where $\overline{1} = (1)_{i \in J}$, and for $(\widetilde{f}, \overline{f}), (\widetilde{g}, \overline{g}) \in \mathrm{Aut}(\mathbf{J}) \times L^J$,

$$(\widetilde{f}, \overline{f}) \cdot (\widetilde{g}, \overline{g}) = (\widetilde{f} \circ \widetilde{g}, (\overline{f} \otimes \widetilde{g}) \cdot \overline{g}),$$

$$(\widetilde{f}, \overline{f})^{\ell} = (\widetilde{f}^{-1}, \overline{f}^{\ell} \otimes \widetilde{f}^{-1}) \text{ and } (\widetilde{f}, \overline{f})^{r} = (\widetilde{f}^{-1}, \overline{f}^{r} \otimes \widetilde{f}^{-1}),$$

where for $h \in \mathrm{Aut}(\mathbf{J})$ and $a \in L^J$, $(a \otimes h)_i = a_{h(i)}$;

the order of $\mathbf{Aut}(\mathbf{J}) \wr \mathbf{L}$ is defined by

$$(\widetilde{f}, \overline{f}) \leq (\widetilde{g}, \overline{g}) \iff \widetilde{f} \leq \widetilde{g} \text{ and } (\widetilde{f}(i) = \widetilde{g}(i) \implies \overline{f}_i \leq \overline{g}_i, \text{ for all } i \in J).$$

- If $\mathbf{L}$ is an n-periodic ℓ-pregroup, then $\mathbf{Aut}(\mathbf{J}) \wr \mathbf{L}$ is an n-periodic ℓ-pregroup.

Local subalgebra

- Every n-periodic ℓ-pregroup embeds into $\mathbf{F}_n(\mathbf{J} \overrightarrow{\times} \mathbb{Z}) \cong \mathbf{Aut}(\mathbf{J}) \wr \mathbf{F}_n(\mathbb{Z})$ for some chain $\mathbf{J}$ (Galatos-Gallardo 2025).

Fact. For $\mathbf{L} \leq \mathbf{Aut}(\mathbf{J}) \wr \mathbf{F}_n(\mathbb{Z})$ the set $L_{\text{loc}} = \{(\widetilde{f}, \overline{f}) \in L : \widetilde{f} = id\}$ forms a subalgebra, which we call the **local subalgebra** of $\mathbf{L}$.

- The local subalgebra depends on the representation.

Lem. Let $\mathbf{L} \leq \mathbf{Aut}(\mathbf{J}) \wr \mathbf{F}_n(\mathbb{Z})$.

(1) $\mathbf{L}_{\text{loc}}$ is a convex normal subalgebra of $\mathbf{L}$.

(2) $(\mathbf{L}_g)_{\text{loc}} = (\mathbf{L}_{\text{loc}})_g$ is a convex normal subalgebra of $\mathbf{L}_g$.

(3) $\mathbf{L}/\mathbf{L}_{\text{loc}} \cong \mathbf{L}_g/(\mathbf{L}_g)_{\text{loc}}$.

(4) If $\mathbf{L}_g \neq \mathbf{L}$ is totally ordered, then $\mathbf{L}_{\text{loc}} \cong \mathbf{F}_k(\mathbb{Z})$ for $k \mid n$ and $(\mathbf{L}_g)_{\text{loc}} \cong \mathbb{Z}$.

- Item (4) relies on the result that $\mathbf{F}_n(\mathbb{Z})$ is generated by any of its elements of periodicity n. (sizable part of our paper!)

Characterization of finitely generated FSIs

Lem. If $\mathbf{G}$ is a finitely generated totally ordered abelian group with convex subgroup $\mathbb{Z}$, then $\mathbf{G} \cong \mathbf{G}(\mathbb{Z}) \overrightarrow{\times} \mathbb{Z}$.

Lem. If $\mathbf{L} \leq \mathbf{Aut}(\mathbf{J}) \wr \mathbf{F}_n(\mathbb{Z})$ is a finitely generated n-periodic ℓ-pregroup with $\mathbf{L}_g \neq \mathbf{L}$ totally ordered and abelian, then $\mathbf{L}_g$ is finitely generated and $\mathbf{L}_g \cong \mathbf{L}_g/(\mathbf{L}_g)_{\mathrm{loc}} \overrightarrow{\times} \mathbb{Z}$.

Thm. If $\mathbf{L} \leq \mathbf{Aut}(\mathbf{J}) \wr \mathbf{F}_n(\mathbb{Z})$ is a finitely generated n-periodic ℓ-pregroup with $\mathbf{L}_g \neq \mathbf{L}$ totally ordered and abelian, then $\mathbf{L} \cong \mathbf{H} \overrightarrow{\times} \mathbf{F}_k(\mathbb{Z})$ with $k \mid n$ and $\mathbf{H}$ a finitely generated totally ordered abelian ℓ-group.

Proof idea. We know $\mathbf{L}_g \cong \mathbf{G} \overrightarrow{\times} \mathbb{Z}$ and $(\mathbf{L}_{\mathrm{loc}})_g$ '$=$' $\mathbb{Z}$.

- ▷ Find an isomorphic copy $\mathbf{H}$ of $\mathbf{G}$ in $\mathbf{L}_g$ such that its elements commute with the elements of $\mathbf{L}_{\mathrm{loc}}$.
- ▷ Show that $\mathbf{L} \cong \mathbf{H} \overrightarrow{\times} \mathbf{L}_{\mathrm{loc}}$.

Conclusion

We showed:

- $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is axiomatized relative to LP_n by $\sigma_n(x)^n y \approx y\sigma_n(x)^n$.
- The FSIs of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ are exactly the n-periodic ℓ-pregroups with a totally ordered abelian group skeleton.
- The finitely generated FSIs of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ are of the form $\mathbf{H} \overrightarrow{\times} \mathbf{F}_k(\mathbb{Z})$ with $k \mid n$ and $\mathbf{H}$ a finitely generated totally ordered abelian ℓ-group.

We further obtain:

- Explicit finite axiomatizations for (finite) joins of varieties of the form $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ and explicit descriptions of their subvariety lattices.
- $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ has only finitely many subvarieties. So it is 'small' in the subvariety lattice of LP_n.

Work in progress: Every proper subvariety of DLP is periodic.

Thank you!

For more details see:

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