# New Old Results

Ralph Freese

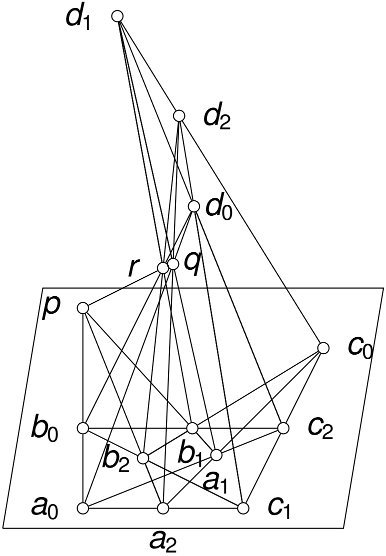
http://math.hawaii.edu/∼ralph/

http://uacalc.org/

https://github.com/UACalc/

http://math.hawaii.edu/∼ralph/Day/

Panglobal Algebra and Logic Seminar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- No (Mark Haiman)

# A Universal Algebra Result

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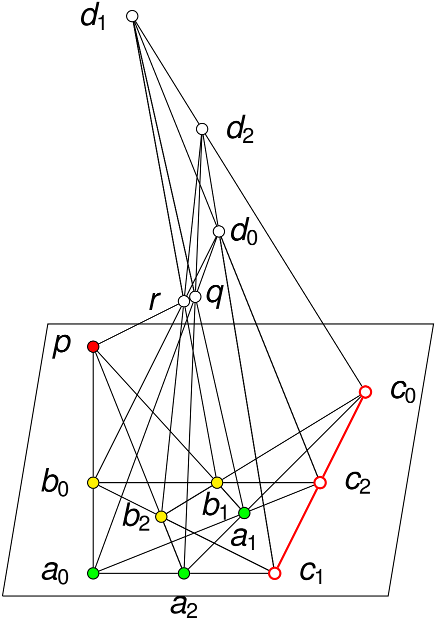
- Desagues Law holds in higher dimensional spaces.
- Increase the dimension using $\boldsymbol{S}$ and $\boldsymbol{P}$.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# Higher arguesian identities: Bill Lampe

- A chain of identities of increasing strength:

$$\bigwedge_{i=0}^{n-1}(x_i \vee x_i') \leq x_0' \vee (x_0 \wedge (x_1 \vee [(x_0' \vee x_1') \wedge \bigvee_{i=1}^{n-1} y_i])) \qquad (*_n)$$

where $y_i = (x_i \vee x_{i+1}) \wedge (x_i' \vee x_{i+1}')$, mod $n$ so $y_{n-1} = (x_{n-1} \vee x_0) \wedge (x_{n-1}' \vee x_0')$.

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- Diagram showing $(*_n)$ holds in lattices of perm. equiv. rels.

$$\bigwedge_{i=0}^{n-1}(\alpha_i \vee \alpha_i') \leq \alpha_0' \vee (\alpha_0 \wedge (\alpha_1 \vee [(\alpha_0' \vee \alpha_1') \wedge \bigvee_{i=1}^{n-1} \gamma_i])) \qquad (*_n)$$

where $\gamma_i = (\alpha_i \vee \alpha_{i+1}) \wedge (\alpha_i' \vee \alpha_{i+1}')$, mod $n$ so
$\gamma_{n-1} = (\alpha_{n-1} \vee \alpha_0) \wedge (\alpha_{n-1}' \vee \alpha_0')$.

Let $\langle a, b\rangle \in \bigwedge_{i=0}^{n-1}(\alpha_i \vee \alpha_i') = \bigwedge_{i=0}^{n-1}(\alpha_i \circ \alpha_i')$. Then there exists $c_i$ so that

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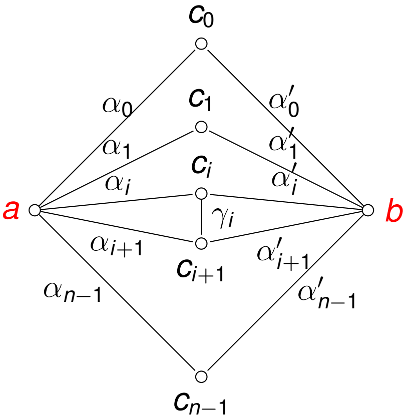

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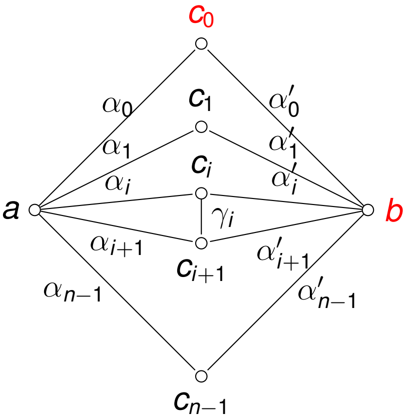

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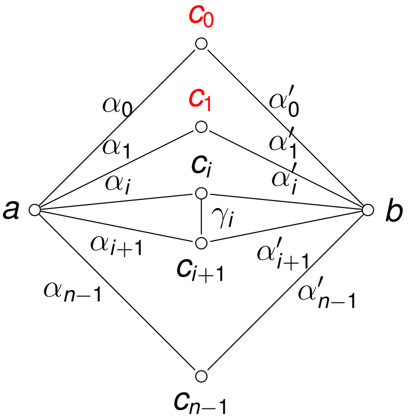

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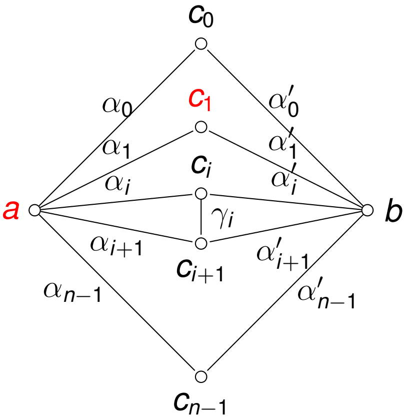

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**Remark:** The **relation** $(*_n)$ holds if $\alpha_i$ and $\alpha_i'$ permute, for each $i$.

# Haiman's Lattices

Are the equations $(*_n)$ **properly** increasing in strength?

Haiman constructs lattices $\mathbf{H}_n(\mathbf{F})$, $n \geq 3$ and $\mathbf{F}$ a field with $|\mathbf{F}| > 2$, such that

- $(*_n)$ fails in $\mathbf{H}_n(\mathbf{F})$.
- Every $n - 1$ generated sublattice is proper.
- Every proper sublattice is embeddable into the lattice of subspaces of a vector space over $\mathbf{F}$.

**Theorem (Haiman)**

*The class of lattices of permuting equivalence relations is not finitely axiomatizable.*

- Notation: $\mathbf{Con}(\mathcal{V}) := \{\mathbf{Con}(\mathbf{A}) : \mathbf{A} \in \mathcal{V}\}$.
- For $\mathcal{V}$ a variety define the
  - **congruence prevariety:** $\boldsymbol{SP}\,\mathbf{Con}(\mathcal{V}) = \boldsymbol{S}\,\mathbf{Con}(\mathcal{V})$
  - **congruence variety:** $\boldsymbol{HSP}\,\mathbf{Con}(\mathcal{V}) = \boldsymbol{HS}\,\mathbf{Con}(\mathcal{V})$

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**Theorem (RF, 1994)**

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**Theorem (RF, 1994)**

*If a modular congruence variety is finitely based, then it is distributive.*

**Theorem (P. Lipparini & RF, real soon)**

*If a proper congruence variety is finitely based, then it is join semidistributive.*

# Elements of the Proof

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- A nonprincipal ultraproduct of the $\mathbf{H}_n(\mathbf{F})$'s ($\mathbf{F}$ fixed) lies in a lattice of subspaces of a vector space over $\mathbf{F}$.
- If $\mathcal{V}$ is not congruence join semidistribuitve then its congruence variety $\boldsymbol{V}\,\mathbf{Con}(\mathcal{V})$ contains $\boldsymbol{V}\,\mathbf{Con}(\mathcal{M}_p)$ for some $p$, a prime or 0.

# Elements of the Proof

- Let

$$\mathcal{C}_\infty = \bigcap_{p \text{ a prime or } 0} \boldsymbol{S}\,\mathbf{Con}(\mathcal{M}_p^{\mathrm{fd}}).$$

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**Corollary**

*If $\mathcal{V}$ is a variety with a weak difference term but which is not congruence meet semidistributive, then $\mathcal{C}_\infty \subseteq \mathbf{S}\,\mathbf{Con}(\mathcal{V})$.* □

# Elements of the Proof

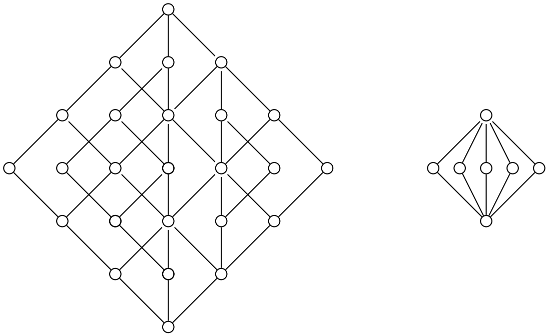

Figure: Two members of $\mathcal{C}_\infty$

# Mal'tsev Conditions and the Commutator

Related Mal'tsev conditions:

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- If $\mathcal{V}$ has a Hobby-McKenzie term then its congruence variety does not contain $\mathbf{H}_n(\mathbf{F})$. But also
- If $\mathcal{V}$ has a weak difference term then its congruence prevariety does not contain $\mathbf{H}_n(\mathbf{F})$.

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We do know

- $\mathfrak{K}$ is a join subsemilattice of $\mathfrak{L}$ (for finite joins).
- Infinite joins can differ.

# Alan Day: Higher Polin Varieties


Alan Day.
Polin's Non-Modular Congruence Variety.
14 pages, 1977.
math.hawaii.edu/∼ralph/Day/

Alan Day.
Polin's Non-Modular Congruence Variety–Corrigendum.
5 pages, 1977.
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Alan Day, Ralph Freese.
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- Alan Day, Ralph Freese, J. B. Nation.
  Higher Polin Varieties.
  Coming soon.

# Higher Polin Varieties

Alan Day  Ralph Freese  and J. B. Nation

**Abstract.** A chain of varieties based on Polin's variety are introduced and studied. It is shown that the collection of all congruence varieties has finite joins and that this join agrees with the join in the lattice of all lattice varieties. Using the higher Polin varieties it is shown that infinite joins may di er from the join in the lattice of all varieties. Other properties of higher Polin varieties are explored.

**Mathematics Subject Classi cation.** 06B20, 08A30, 08B05, 08B15.

**Keywords.** congruence lattice, congruence variety, congruence semidistributivity,  -permutability.

Let $\mathcal{V}$ be a variety of algebras and let

$$\mathbf{Con}(\mathcal{V}) = \{\mathbf{Con}(\mathbf{A}) : \mathbf{A} \in \mathcal{V}\}.$$

The variety of lattices $\boldsymbol{V}\mathbf{Con}(\mathcal{V}) = \quad \boldsymbol{SP}\mathbf{Con}(\mathcal{V})$ generated by the congruence lattices of the members of $\mathcal{V}$, is called the *congruence variety* associated with $\mathcal{V}$. Early in our careers the three of us studied congruence varieties. The subject began with the third author's thesis [16] which showed, among other things, that the lattice variety generated by $\mathbf{N}_5$ (the 5 element nonmodular lattice) is not a congruence variety; see [7, Theorem 6.99]. This led to several papers, which generalized Nations's result, by us and others, [1, 4, 9, 10]. The major open problem of the time was known alternately as the McKenzie conjecture, the Burris-McKenzie conjecture and Jónsson's question. It asked if there is a nonmodular congruence variety other than the variety of all lattices. This was resolved by S. V. Polin [17] who constructed a variety $\mathcal{P}$ which is not congruence modular but does satisfy a nontrivial lattice equation as a congruence identity.

Communication between Russia and the west was not ideal in those days and preprints of Polin's paper were not available. Pavel Goralčík was able to obtain the details from Polin and communicated a sketch to the  rst author in Prague. In 1977 the  rst author lectured on the result at Vanderbilt and wrote up notes, [2, 3]. Studying these notes carefully led to the paper [4] by the  rst two authors. It shows that any nonmodular congruence variety

Corresponding author.

# Alan Day: Higher Polin Varieties

- $\mathcal{B}$ is the variety of Boolean algebras, signature $\{\wedge, 1, '\}$
- $\mathcal{Q}$ a variety of meet semilattices with constant 1 and some unary operations $u_i$.
- Let $\mathbf{A} \in \mathcal{B}$ and $\mathbf{S}$ a functor from $\mathbf{A}$ to $\mathcal{Q}$. So for $a \geq b \in A$
  - $\mathbf{S}(a) \in \mathcal{Q}$ and
  - $\xi_b^a : \mathbf{S}(a) \to \mathbf{S}(b)$ are compatible homomorphisms.
- Let $\mathcal{Q}[\mathcal{B}]$ have members:

$$\mathbf{P}(\mathbf{S}, \mathbf{A}) = \bigcup_{a \in A} \{a\} \times \mathbf{S}(a) \quad \text{with}$$

$$\begin{aligned} \langle a, s\rangle \wedge \langle b, t\rangle &= \langle a \wedge b, \xi_{a\wedge b}^a(s) \wedge \xi_{a\wedge b}^b(t)\rangle \\ 1 &= \langle 1, 1\rangle \\ f^{\mathbf{P}}(\langle a, s\rangle) &= \langle a, f^{\mathbf{S}(a)}(s)\rangle, \quad f \text{ unary in the signature of } \mathcal{Q} \\ \langle a, s\rangle^+ &= \langle a', 1\rangle \qquad \text{(external complement)} \end{aligned}$$

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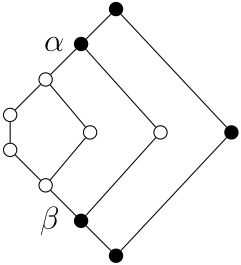
- $\mathcal{P}_1$ is Boolean algebras and $\mathcal{P}_2$ is Polin's variety.
- $\mathbf{N}_5$, $\mathbf{N}_6$, etc. be McKenzie lattices:




$\mathbf{N}_7$

# Alan Day: Higher Polin Varieties

## Theorem

*For* $k \geq 2$, $\mathbf{N}_{k+3} \in \boldsymbol{S}\mathbf{Con}(\mathcal{P}_k)$ *but* $\mathbf{N}_{k+4} \notin \boldsymbol{V}\mathbf{Con}(\mathcal{P}_k)$*. The containments*

$$\boldsymbol{V}\mathbf{Con}(\mathcal{P}_1) \subset \boldsymbol{V}\mathbf{Con}(\mathcal{P}_2) \subset \boldsymbol{V}\mathbf{Con}(\mathcal{P}_3) \subset \cdots \qquad (*)$$

*form a strictly increasing chain of congruence varieties whose set union, which is the join in the lattice* $\mathfrak{L}$ *of lattice varieties, is not a congruence variety.*

# Alan Day: Higher Polin Varieties

## Theorem

*If $\mathcal{Q}$ is $k$-permutable, then $\mathcal{Q}[\mathcal{B}]$ is $(k+2)$-permutable.*

## Corollary

*$\mathcal{P}_n$ is $2n$-permutable.*

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**Theorem (using Kearnes-Nation 2008)**

*If $\mathcal{V}$ is congruence meet semidistributive and n-permutable then its whole congruence variety is semidistributive (both kinds).*


MATH