Generic automorphisms of Boolean powers

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Finite direct powers

Let **A** be a finite simple Mal'cev algebra

- **simple:**
  the total congruence and $=$ are the only congruences of **A**
- **Mal'cev:**
  A has a term m satisfying $m(x, y, y) = x = m(y, y, x)$
- **abelian:**
  operations of **A** are affine on a module

Examples

abelian

- simple modules $F^{n\times n}$-module F^n
- $(\mathbb{Z}_p, x - y + z, x + 1)$

non-abelian

- simple non-abelian (quasi)groups $(A_5, \cdot)$
- full matrix rings over finite fields $(F^{n\times n}, +, \cdot)$

Homomorphisms between finite direct powers

Lemma

Let $\mathbf{A}$ be a finite simple Mal'cev algebra, $k, n \in \mathbb{N}$, and $h\colon \mathbf{A}^k \to \mathbf{A}^n$ a homomorphism.

1. If $\mathbf{A}$ is abelian, then

$$h(x_1, \dots, x_k) = (x_1, \dots, x_k) \cdot B + c$$

   for some field F, $B \in F^{k \times n}$ and $c \in A^n$.

2. (Foster, Pixley) If $\mathbf{A}$ is non-abelian, then

$$h(x_1, \dots, x_k) = (\alpha_1(x_{i_1}), \dots, \alpha_n(x_{i_n}))$$

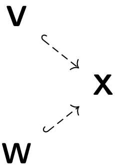
   for $\alpha_1, \dots, \alpha_n$ automorphisms or endomorphisms to a trivial subalgebra of $\mathbf{A}$.

A little bit of Fraïssé

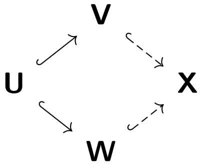
Let $\mathbf{A}$ be a finite simple Mal'cev algebra.

$K := \{\mathbf{A}^k \mid k \in \mathbb{N}\}$ has

- the **joint embedding property (JEP)**,



- the **amalgamation property (AP)**
  iff $\mathbf{A}$ is non-abelian
  or has a trivial subalgebra,



- the **hereditary property (HP)**
  iff all proper subalgebras of $\mathbf{A}$ are trivial.

(Generalized) Fraïssé limit

Theorem (cf. Fraïssé)

Let K be a countable class of finite structures with JEP and AP. There exists a unique (up to isomorphism) countable structure $\mathbf{D} =: \mathrm{Flim} K$, the generalized **Fraïssé limit** of K, such that

1. every finitely generated substructure of $\mathbf{D}$ embeds into some element of K,
2. $\mathbf{D}$ is a direct limit of structures in K,
3. every isomorphism between substructures of $\mathbf{D}$ that are in K extends to an automorphism of $\mathbf{D}$ ($\mathbf{D}$ is K-homogeneous).

Question

What is $\mathrm{Flim}\{\mathbf{A}^k \mid k \in \mathbb{N}\}$ for a finite simple Mal'cev algebra $\mathbf{A}$?

A little bit of Ramsey

Theorem (M, Ruškuc 2025)

Let **A** be a finite simple non-abelian Mal'cev algebra. Then

$$K := \{\mathbf{A}^m \mid m \in \mathbb{N}\}$$

is **Ramsey**,
i.e. for all $k, m, n \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that for every k-coloring of copies of $\mathbf{A}^m$ in $\mathbf{A}^r$ there is a monochromatic copy of $\mathbf{A}^n$ in $\mathbf{A}^r$.

Proof.
Follows from the Foster-Pixley Theorem and the Graham-Rothschild Theorem. □

Hence the automorphism group of $\mathrm{Flim}\, K$ with a natural order is **extremely amenable** (Kechris, Pestov, Todorcevic 2005).

Question

What is $\mathrm{Flim}\, K$?

Boolean powers

Filtered Boolean powers (Arens, Kaplansky 1948)

A finite algebra (with discrete topology),
B Boolean algebra with Stone space X.

$$\mathbf{A}^{\mathbf{B}} := \{f \colon X \to A \mid f \text{ continuous}\} \leq \mathbf{A}^X$$

is a **Boolean power**.

e is **idempotent** if $\{e\} \leq \mathbf{A}$.

For distinct $x_1, \ldots, x_n \in X$ and idempotents $e_1, \ldots, e_n$ in **A**,

$$(\mathbf{A}^{\mathbf{B}})^{x_1,\ldots,x_n}_{e_1,\ldots,e_n} := \{f \in \mathbf{A}^{\mathbf{B}} \mid f(x_i) = e_i \text{ for all } i\} \leq \mathbf{A}^X$$

is a **filtered Boolean power**.

The Fraïssé limit as filtered Boolean power

Theorem (M, Ruškuc 2023)

For a finite simple Mal'cev algebra $\mathbf{A}$, non-abelian or with trivial subalgebra,

$$\mathrm{Flim}\{\mathbf{A}^k \mid k \in \mathbb{N}\} \cong (\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}$$

where

- $e_1, \dots, e_n$ is the set of all idempotents of $\mathbf{A}$,
- $\mathbf{B}$ is the countable atomless Boolean algebra with distinct $x_1, \dots, x_n$ in its Stone space 2^ω (the Cantor space).

Proof.

$(\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}$ satisfies the defining properties of the Fraïssé limit. □

ω-categorical filtered Boolean powers

A countable structure M is ω-**categorical** if its theory has a unique countable model (up to isomorphism).

Theorem (cf. Macintyre, Rosenstein 1976)

Let $\mathbf{A}$ be a finite algebra, $\mathbf{B}$ the countable atomless Boolean algebra.
Then any filtered Boolean power $(\mathbf{A}^{\mathbf{B}})^{x_1,\ldots,x_n}_{e_1,\ldots,e_n}$ is ω-categorical.

Automorphism groups

Largeness in permutation groups

Let M be a countable infinite structure, $G := \mathrm{Aut}\, M$.

- G is a topological (Polish) group under **pointwise convergence**: basic open sets are cosets of stabilizers of finite tuples over M

$$G_{m_1,\ldots,m_k} := \{g \in G \mid g(m_i) = m_i \text{ for all } i \leq k\}.$$

- $(h_1, \ldots, h_n) \in G^n$ is **generic** if its orbit under the diagonal conjugation action of G,

$$\{(h_1^g, \ldots, h_n^g) \mid g \in G\},$$

  is **comeager**, i.e., contains the intersection of countably many dense open subsets of G^n.
- G has **ample generics** if it has generic n-tuples for each $n \in \mathbb{N}$.

Theorem (Kechris, Rosendal 2007)

Assume $G = \mathrm{Aut}\, M$ has ample generics.

1. Then G has the **small index property (SIP)**
   i.e., each $H \leq G$ of index $< 2^{\aleph_0}$ is open;
2. If M is also ω-categorical, then G has
   a. **uncountable cofinality**,
      i.e., G is not a countable union of a chain of proper subgroups,
   b. the **Bergman property**,
      i.e., for each generating set $1 \in E = E^{-1}$ of G there exists $k \in \mathbb{N}$ such that $G = E^k$.

Overview

countably infinite M	SIP	uncountable cofinality	Bergman	ample generics
set $\mathbb{N}$	Dixon Neumann Thomas '86	Macpherson Neumann '86	Bergman '06	Kechris Rosendal '07
random graph	Hodges Hodkinson Lascar Shelah '93	Hodges Hodkinson Lascar Shelah '93	Kechris Rosendal '07	Hrushovsky '92
$(\mathbb{Q}, \leq)$	Truss '89	Gourion '92	Droste Holland '05	no, Hodkinson
vector space over fin field	Evans '86	Thomas '96	Tolstykh '06	???
2^ω, atomless Boolean **B**	Truss '87	Droste Göbel '05	Droste Göbel '05	Kwiatkowska '12
$(\mathbf{A}^\mathbf{B})_{e_1,\dots,e_n}^{x_1,\dots,x_n}$	M Ruškuc '23	M Ruškuc '23	M Ruškuc '23	M Ruškuc '25

Automorphism groups of filtered Boolean powers

Automorphisms of a Boolean algebra $\mathbf{B}$ correspond to **homeomorphisms** (continuous bijections) of its Stone space X,

$$\mathrm{Aut}\ \mathbf{B} \cong \mathrm{Homeo}\ X.$$

Theorem (M, Ruškuc 2023; for groups cf. Apps 1981)

Let $\mathbf{A}$ be a finite simple non-abelian Mal'cev algebra with idempotents $e_1, \ldots, e_n$ in distinct $\mathrm{Aut}\ \mathbf{A}$-orbits,
$\mathbf{B}$ the countable atomless Boolean algebra,
2^ω the Cantor space with distinct $x_1, \ldots, x_n \in 2^\omega$. Then

$$\mathrm{Aut}\left((\mathbf{A}^\mathbf{B})^{x_1,\ldots,x_n}_{e_1,\ldots,e_n}\right) \cong N \rtimes (\mathrm{Homeo}\, 2^\omega)_{x_1,\ldots,x_n}$$

where N is isomorphic to the closure of $((\mathrm{Aut}\,\mathbf{A})^\mathbf{B})^{x_1,\ldots,x_n}_{1,\ldots,1}$.

Ample generics for pointwise stabilizers in $\mathrm{Homeo}\, 2^\omega$

Theorem (M, Ruškuc 2025; for $n = 0$ Kwiatkowska 2012)
$H := (\mathrm{Homeo}\, 2^\omega)_{x_1,\ldots,x_n}$ for $x_1, \ldots, x_n \in 2^\omega$ has ample generics.

Proof idea
$(h_1, \ldots, h_m)$ is a generic m-tuple in H^m iff

$$(2^\omega, h_1, \ldots, h_m, x_1, \ldots, x_n)$$

is a **projective Fraïssé limit** (Irwin, Solecki 2006) of finite topological structures

$$(A, s_1, \ldots, s_m, p_1, \ldots, p_n)$$

for certain binary relations $s_1, \ldots, s_m$ and constants $p_1, \ldots, p_n$.

Ample generics for filtered Boolean powers

Theorem (M, Ruškuc 2025)

Let $\mathbf{A}$ be a finite simple nonabelian Mal'cev algebra,
$\mathbf{B}$ the countable atomless Boolean algebra.
Then $G := \mathrm{Aut}\left((\mathbf{A}^{\mathbf{B}})^{x_1,\dots,x_n}_{e_1,\dots,e_n}\right)$ has ample generics.

Proof.

- Recall $G = N \rtimes H$
  for N isomorphic to the closure of $((\mathrm{Aut}\,\mathbf{A})^{\mathbf{B}})^{x_1,\dots,x_n}_{1,\dots,1}$
  and $H \cong (\mathrm{Homeo}\,X)_{x_1,\dots,x_n}$.
- Let $\mathbf{h}$ be a generic m-tuple in H^m.
- For every $\mathbf{a} \in N^m$ there exists $c \in N$ such that $\mathbf{ah} = \mathbf{h}^c$.
- Then $\mathbf{h}^G = N^m\mathbf{h}^H$ is dense and the intersection of countably many open subsets of G.
- Hence $\mathbf{h}$ is a generic m-tuple in G^m.

□

Questions

Open

Let $\mathbf{A}$ be a finite simple non-abelian group (Mal'cev algebra).

Question

Is every $\mathbf{C}$ in the variety $\mathrm{HSP}(\mathbf{A})$ generated by $\mathbf{A}$ an extension of a filtered Boolean power of $\mathbf{A}$ by a group in HSP(proper subgroups of $\mathbf{A}$)?

True for countable $\mathbf{C}$.

Question (Bryant, Evans 1997)

Does the automorphism group of the free group of countable rank in $\mathrm{HSP}(\mathbf{A})$ have the small index property?

Let **A** be a finite 1-dim vector space (abelian Mal'cev algebra), **B** the countable atomless Boolean algebra.
Then $\mathbf{V} := \mathbf{A}^{\mathbf{B}} = \mathrm{Flim}\,\{\mathbf{A}^k \mid k \in \mathbb{N}\}$ is the vector space of countable dimension.

Question

Does $\mathrm{Aut}\,\mathbf{A}^{\mathbf{B}} = \mathrm{GL}\,\mathbf{V}$ have ample generics?

Known that $\mathrm{GL}\,\mathbf{V}$ has SIP, uncountable cofinality, Bergman property, $\mathcal{B}(\mathbf{V})$-generics (Hodges, Hodkinson, Lascar, Shelah 1993).